Number Theory Problems from IMO Shortlist 1999 – 2006

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There'll be plenty of time to rest in the grave. - Paul Erdös

Contents

Chapter 1

47th IMO Slovenia 2006

1 Determine all pairs (x, y) of integers such that

06N01
IMO 06/4

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution

2 For $x \in (0,1)$ let $y \in (0,1)$ be the number whose n-th digit after the decimal point is the 2^n-th digit after the decimal point of x. Show that if x is rational then so is y.

06N02

SOLUTION

3 We define a sequence $(a_1, a_2, a_3, ...)$ by setting

06N03
Italy '07

$$a_n = \frac{1}{n}\left(\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right]\right)$$

for every positive integer n.

a) Prove that there is an infinite number of positive integers n such that $a_{n+1} > a_n$.

b) Prove that there is an infinite number of positive integers n such that $a_{n+1} < a_n$.

SOLUTION

4
06N04

Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\cdots P(P(x))\cdots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

SOLUTION

5 Prove that the equation $\frac{x^7-1}{x-1} = y^5 - 1$ does not have integer solutions.

06N05
Brazil '07

SOLUTION

6 06N06 Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer c, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that $ax + by = c$. An integer c is called a local champion if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$. Find all local champions and determine their number.

SOLUTION

7 For all positive integers n, show that there exists a positive integer m such that n divides $2^m + m$.

06N07

REMARK. (Brazil '05) Given positive integers a, c and integer b, prove that there exists a positive integer x such that $a^x + x \equiv b \ (mod\ c)$.

SOLUTION

Chapter 2

46th IMO Mexico 2005

8 Determine all positive integers relatively prime to all the terms of the infinite sequence

05N01
IMO 05/4

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

Solution

9
05N02
IMO 05/2

Let $a_1, a_2, \cdots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \cdots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \cdots$.

Solution

10

05N03
India '06

Let a, b, c, d, e, f be positive integers and let $S = a + b + c + d + e + f$. Suppose that the number S divides $abc + def$ and $ab + bc + ca - de - ef - df$. Prove that S is composite.

SOLUTION

11 Find all n such that there exists a unique integer a such that $0 \leq a < n!$ with the following property:

05N04
Iran '06

$$n! \mid a^n + 1.$$

Solution

12
05N05

Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called highly divisible if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no highly divisible integer s satisfying $m < s < n$.

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a, b) with $a \mid b$.

(b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.

SOLUTION

13 Let a, b be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers n. Prove that $a = b$.

05N06
Taiwan '06

SOLUTION

14 05N07 Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0, \cdots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.

Solution

Chapter 3

45th IMO Greece 2004

15
04N01

Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

Solution

16

04N02

The function f from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$f(n) = \sum_{k=1}^{n} \gcd(k, n), \qquad n \in \mathbb{N}.$$

a) Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.

b) Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.

c) Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

SOLUTION

17 Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

04N03

$$f(m)^2 + f(n) \mid \left(m^2 + n\right)^2$$

for all positive integers m and n.

SOLUTION

18 Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

04N04
Poland '05

$$x_0 = a,\ x_1 = b,\ x_{n+2} = x_{n+1} + x_n, n = 0, 1, 2, \cdots$$

has all of its terms relatively prime to m.

SOLUTION

19

04N05

IMO 04/6

We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.

SOLUTION

20 04N06 Taiwan '05

Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

Solution

21

04N07

Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

SOLUTION

Chapter 4

44th IMO Japan 2003

22 Let $x_0, x_1, x_2, \cdots$ be the sequence defined by

03N01
Singapore
'04

$$x_i = 2^i,\ 0 \le i \le 2003,$$

$$x_i = \sum_{j=1}^{2004} x_{i-j},\ i \ge 2004.$$

Find the greatest k for which the sequence contains k consecutive terms divisible by 2004.

Solution

23

03N02
Germany
'04

Each positive integer a is subjected to the following procedure, yielding the number $d = d(a)$: (a) The last digit of a is moved to the first position. The resulting number is called b. (b) The number b is squared. The resulting number is called c. (c) The first digit of c is moved to the last position. The resulting number is called d. (All numbers are considered in the decimal system.) For instance, $a = 2003$ gives $b = 3200$, $c = 10240000$ and $d = 02400001 = 2400001 = d(2003)$. Find all integers a such that $d(a) = a^2$.

SOLUTION

24 Determine all pairs of positive integers (a, b) such that

03N03
IMO 03/2

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

SOLUTION

25 03N04 Germany '04

Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b. Prove that the following condition holds if and only if $b = 10$: there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

SOLUTION

26

03N05
Moldova
'04

An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

SOLUTION

27 Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

03N06
IMO 06/6

SOLUTION

28
03N07

The sequence a_0, a_1, a_2, $\cdots$ is defined as follows:

$$a_0 = 2, \ \ a_{k+1} = 2a_k^2 - 1, \ k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

SOLUTION

29
03N08

Let p be a prime number and let A be a set of positive integers that satisfies the following conditions: (1) the set of prime divisors of the elements in A consists of $p-1$ elements; (2) for any nonempty subset of A, the product of its elements is not a perfect p-th power. What is the largest possible number of elements in A ?

SOLUTION

Chapter 5

43rd IMO United Kingdom 2002

30 What is the smallest positive integer t such that there exist integers $x_1, x_2, \cdots, x_t$ with

02N01

$$x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002}\,?$$

Solution

31 Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that
02N02 $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor
IMO 02/4 of n^2.

SOLUTION

32 Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.

02N03

SOLUTION

33 02N04

Is there a positive integer m such that the equation

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{abc}=\frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c?

Solution

34 Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \cdots, a_n$ be integers, none of which is
02N05 a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \cdots, e_n$, not all zero, with $|\, e_i \,| < m$ for all i, such that $e_1 a_1 + e_2 a_2 + \cdots + e_n a_n$ is a multiple of m^n.

SOLUTION

35 Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

02N06
IMO 02/3

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

SOLUTION

Chapter 6

42nd IMO United States of America 2001

36 Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.

01N01

SOLUTION

37
01N02

Consider the system $x + y = z + u$, $2xy = zu$. Find the greatest value of the real constant m such that $m \leq x/y$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

SOLUTION

38 Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and $a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4$.

01N03 Determine $a_{14^{14}}$.

SOLUTION

39 Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$
01N04 such that neither $a^{p-1} - 1$ nor $(a+1)^{p-1} - 1$ is divisible by p^2.

SOLUTION

40 Let $a > b > c > d$ be positive integers and suppose that

01N05
IMO 01/6

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

SOLUTION

41
01N06

Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?

SOLUTION

Chapter 7

41st IMO Korea 2000

42 Determine all positive integers $n \geq 2$ that satisfy the following condition: For all integers a and b relatively prime to n,

00N01

$$a \equiv b \ (mod\ n)$$

if and only if

$$ab \equiv 1 \ (mod\ n).$$

SOLUTION

43 For every positive integers n let $d(n)$ the number of all positive integers of n. Determine all positive integers n with the property: $d^3(n) = 4n$.

00N02

Solution

44 00N03 IMO 01/5

Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

SOLUTION

45 Find all triplets of positive integers (a, m, n) such that $a^m + 1 \mid (a+1)^n$.

00N04

SOLUTION

46
00N05

Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

SOLUTION

47

00N06

Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.

Solution

Chapter 8

40th IMO Romania 1999

48 Find all the pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

99N01
IMO 99/4

SOLUTION

49 Prove that every positive rational number can be represented in the form $\frac{a^3+b^3}{c^3+d^3}$ where a,b,c,d are positive integers.

99N02

SOLUTION

50 Prove that there exists two strictly increasing sequences (a_n) and (b_n) such that
99N03 $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural n.

SOLUTION

51
99N04

Denote by S the set of all primes such the decimal representation of $\frac{1}{p}$ has the fundamental period divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0, a_1 a_2 \cdots a_{3r} a_1 a_2 \cdots a_{3r} \cdots ,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k, p)$ by

$$f(k, p) = a_k + a_{k+r(p)} + a_{k+2.r(p)}$$

a) Prove that S is infinite. b) Find the highest value of $f(k, p)$ for $k \geq 1$ and $p \in S$.

Solution

52
99N05

Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

SOLUTION

53 Prove that for every real number M there exists an infinite arithmetic progression such that:

99N06

(1) each term is a positive integer and the common difference is not divisible by 10;

(2) the sum of the digits of each term (in decimal representation) exceeds M.

SOLUTION

Chapter 9

All Problems

Problem 1 (06N01 - IMO 06/4). *Determine all pairs (x, y) of integers such that*

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 2 (06N02). *For $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose n-th digit after the decimal point is the 2^n-th digit after the decimal point of x. Show that if x is rational then so is y.*

Problem 3 (06N03 - Italy '07). *We define a sequence $(a_1, a_2, a_3, ...)$ by setting*

$$a_n = \frac{1}{n}\left(\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right]\right)$$

for every positive integer n. Hereby, for every real x, we denote by $[x]$ the integral part of x (this is the greatest integer which is $\leq x$).

a) Prove that there is an infinite number of positive integers n such that $a_{n+1} > a_n$.

b) Prove that there is an infinite number of positive integers n such that $a_{n+1} < a_n$.

Problem 4 (06N04). *Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\cdots P(P(x))\cdots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.*

Problem 5 (06N05 - Brazil '07). *Prove that the equation $\frac{x^7-1}{x-1} = y^5 - 1$ does not have integer solutions.*

Problem 6 (06N06). *Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer c, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that $ax + by = c$. An integer c is called a local champion if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$. Find all local champions and determine their number.*

Problem 7 (06N07). *For all positive integers n, show that there exists a positive integer m such that n divides $2^m + m$.*

REMARK. *(Brazil '05) Given positive integers a, c and integer b, prove that there exists a positive integer x such that $a^x + x \equiv b \pmod{c}$.*

Problem 8 (05N01 - IMO 05/4). *Determine all positive integers relatively prime to all the terms of the infinite sequence*

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1.$$

Problem 9 (05N02 - IMO 05/2). *Let $a_1, a_2, \cdots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \cdots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \cdots$.*

Problem 10 (05N03 - India '06). *Let a, b, c, d, e, f be positive integers and let $S = a + b + c + d + e + f$. Suppose that the number S divides $abc + def$ and $ab + bc + ca - de - ef - df$. Prove that S is composite.*

Problem 11 (05N04 - Iran '06). *Find all n such that there exists a unique integer a such that $0 \leq a < n!$ with the following property:*

$$n! \mid a^n + 1.$$

Problem 12 (05N05). *Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called highly divisible if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no highly divisible integer s satisfying $m < s < n$.*

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a, b) with $a \mid b$.

(b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.

Problem 13 (05N06 - Taiwan '06). *Let a, b be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers n. Prove that $a = b$.*

Problem 14 (05N07). *Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, where $a_0, \cdots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.*

Problem 15 (04N01). *Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.*

Problem 16 (04N02). *The function f from the set $\mathbb{N}$ of positive integers into itself is defined by the equality*

$$f(n) = \sum_{k=1}^{n} \gcd(k, n), \qquad n \in \mathbb{N}.$$

a) Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
b) Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.
c) Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

Problem 17 (04N03). *Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying*

$$f(m)^2 + f(n) \mid \left(m^2 + n\right)^2$$

for all positive integers m and n.

Problem 18 (04N04 - Poland '05). *Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by*

$$x_0 = a,\ x_1 = b,\ x_{n+2} = x_{n+1} + x_n, n = 0, 1, 2, \cdots$$

has all of its terms relatively prime to m.

Problem 19 (04N05 - IMO 04/6). *We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.*

Problem 20 (04N06 - Taiwan '05). *Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.*

Problem 21 (04N07). *Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.*

Problem 22 (03N01 - Singapore '04). *Let $x_0, x_1, x_2, \cdots$ be the sequence defined by*

$$x_i = 2^i,\ 0 \le i \le 2003,$$
$$x_i = \sum_{j=1}^{2004} x_{i-j},\ i \ge 2004.$$

Find the greatest k for which the sequence contains k consecutive terms divisible by 2004.

Problem 23 (03N02 - Germany '04). *Each positive integer a is subjected to the following procedure, yielding the number $d = d(a)$: (a) The last digit of a is moved to the first position. The resulting number is called b. (b) The number b is squared. The resulting number is called c. (c) The first digit of c is moved to the last position. The resulting number is called d. (All numbers are considered in the decimal system.) For instance, $a = 2003$ gives $b = 3200$, $c = 10240000$ and $d = 02400001 = 2400001 = d(2003)$. Find all integers a such that $d(a) = a^2$.*

Problem 24 (03N03 - IMO 03/2). *Determine all pairs of positive integers (a, b) such that*

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Problem 25 (03N04 - Germany '04). *Let b be an integer greater than 5. For each positive integer n, consider the number*

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b. Prove that the following condition holds if and only if $b = 10$: there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

Problem 26 (03N05 - Moldova '04). *An integer n is said to be [i]good[/i] if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.*

Problem 27 (03N06 - IMO 06/6). *Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.*

Problem 28 (03N07). *The sequence $a_0,\ a_1,\ a_2,\ \cdots$ is defined as follows:*

$$a_0 = 2,\ \ a_{k+1} = 2a_k^2 - 1,\ k \ge 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

Problem 29 (03N08). *Let p be a prime number and let A be a set of positive integers that satisfies the following conditions: (1) the set of prime divisors of the elements in A consists of $p-1$ elements; (2) for any nonempty subset of A, the product of its elements is not a perfect p-th power. What is the largest possible number of elements in A ?*

Problem 30 (02N01). *What is the smallest positive integer t such that there exist integers $x_1, x_2, \cdots, x_t$ with*

$$x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002}\,?$$

Problem 31 (02N02 - IMO 02/4). *Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.*

Problem 32 (02N03). *Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n}+1$ has at least 4^n divisors.*

Problem 33 (02N04). *Is there a positive integer m such that the equation*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c?

Problem 34 (02N05). *Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \cdots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \cdots, e_n$, not all zero, with $|\,e_i\,| < m$ for all i, such that $e_1a_1 + e_2a_2 + \cdots + e_na_n$ is a multiple of m^n.*

Problem 35 (02N06 - IMO 02/3). *Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that*

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Problem 36 (01N01). *Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.*

Problem 37 (01N02). *Consider the system $x + y = z + u$, $2xy = zu$. Find the greatest value of the real constant m such that $m \leq x/y$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.*

Problem 38 (01N03). *Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and $a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4$. Determine $a_{14^{14}}$.*

Problem 39 (01N04). *Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1} - 1$ nor $(a+1)^{p-1} - 1$ is divisible by p^2.*

Problem 40 (01N05 - IMO 01/6). *Let $a > b > c > d$ be positive integers and suppose that*

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Problem 41 (01N06). *Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?*

Problem 42 (00N01). *Determine all positive integers $n \geq 2$ that satisfy the following condition: For all integers a and b relatively prime to n,*

$$a \equiv b \ (mod\ n)$$

if and only if

$$ab \equiv 1 \ (mod\ n).$$

Problem 43 (00N02). *For every positive integers n let $d(n)$ the number of all positive integers of n. Determine all positive integers n with the property: $d^3(n) = 4n$.*

Problem 44 (00N03 - IMO 01/5). *Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?*

Problem 45 (00N04). *Find all triplets of positive integers (a, m, n) such that $a^m + 1 \mid (a+1)^n$.*

Problem 46 (00N05). *Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.*

Problem 47 (00N06). *Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.*

Problem 48 (99N01 - IMO 99/4). *Find all the pairs of positive integers* (x, p) *such that* p *is a prime,* $x \leq 2p$ *and* x^{p-1} *is a divisor of* $(p-1)^x + 1$.

Problem 49 (99N02). *Prove that every positive rational number can be represented in the form* $\frac{a^3+b^3}{c^3+d^3}$ *where a,b,c,d are positive integers.*

Problem 50 (99N03). *Prove that there exists two strictly increasing sequences* (a_n) *and* (b_n) *such that* $a_n(a_n+1)$ *divides* b_n^2+1 *for every natural* n.

Problem 51 (99N04). *Denote by* S *the set of all primes such the decimal representation of* $\frac{1}{p}$ *has the fundamental period divisible by 3. For every* $p \in S$ *such that* $\frac{1}{p}$ *has the fundamental period* $3r$ *one may write*

$$\frac{1}{p} = 0, a_1 a_2 \cdots a_{3r} a_1 a_2 \cdots a_{3r} \cdots ,$$

where $r = r(p)$; *for every* $p \in S$ *and every integer* $k \geq 1$ *define* $f(k, p)$ *by*

$$f(k, p) = a_k + a_{k+r(p)} + a_{k+2.r(p)}$$

a) Prove that S *is infinite. b) Find the highest value of* $f(k, p)$ *for* $k \geq 1$ *and* $p \in S$.

Problem 52 (99N05). *Let* n, k *be positive integers such that* n *is not divisible by 3 and* $k \geq n$. *Prove that there exists a positive integer* m *which is divisible by* n *and the sum of its digits in decimal representation is* k.

Problem 53 (99N06). *Prove that for every real number* M *there exists an infinite arithmetic progression such that:*

(1) each term is a positive integer and the common difference is not divisible by 10;

(2) the sum of the digits of each term (in decimal representation) exceeds M.