Number Theory Problems from The William Lowell Putnam Mathematics Competition 1980 – 2006

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THERE'LL BE PLENTY OF TIME TO REST IN THE GRAVE. - PAUL ERDÖS

Problem 1 (06A3). *Let* $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ *be a sequence defined by*

$$x_k = k,\ k = 1, 2, \cdots, 2006$$

and

$$x_{k+1} = x_k + x_{k-2005},\ k \geq 2006.$$

Show that the sequence has 2005 *consecutive terms each divisible by* 2006*.*

Problem 2 (06A5). *Let* n *be a positive odd integer and let* θ *be a real number such that* θ/π *is irrational. Set*

$$a_k = \tan\left(\theta + \frac{k\pi}{n}\right),\ k = 1, 2, \cdots, n.$$

Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

Problem 3 (05A1). *Show that every positive integer is a sum of one or more numbers of the form* $2^r 3^s$*, where* r *and* s *are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)*

Problem 4 (05B1). *Find a nonzero polynomial* $P(x, y)$ *such that* $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ *for all real numbers* a*. (Note:* $\lfloor \nu \rfloor$ *is the greatest integer less than or equal to* ν*.)*

Problem 5 (05B2). *Find all positive integers* $n, k_1, \ldots, k_n$ *such that*

$$k_1 + \cdots + k_n = 5n - 4$$

and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Problem 6 (05B4). *For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.*

Problem 7 (04A3). *Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that*

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \ge 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Problem 8 (04A4). *Show that for any positive integer n, there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form*

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

Problem 9 (04B1). *Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers*

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \cdots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

Problem 10 (03A6). *For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \ne s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?*

Problem 11 (03B3). *Show that for each positive integer n,*

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\left\{1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right\}.$$

(Here, lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\le x$.)

Problem 12 (03B4). *Let*

$$f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$$

where a, b, c, d, e are integers, $a \ne 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \ne r_3 + r_4$, then $r_1 r_2$ is a rational number.

Problem 13 (02A3). *Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \cdots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.*

Problem 14 (02A5). *Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set*

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\}.$$

Problem 15 (02B6). *Let p be a prime number. Prove that the determinant of the matrix*

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

Problem 16 (01A3). *For each integer m, consider the polynomial*

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Problem 17 (01A5). *Prove that there are unique positive integers a, n such that*

$$a^{n+1} - (a + 1)^n = 2001.$$

Problem 18 (01B3). *For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate*

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Problem 19 (00A2). *Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]*

Problem 20 (00A6). *Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.*

Problem 21 (00B1). *Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4N}{7}$ values of j, $1 \leq j \leq N$.*

Problem 22 (00B2). *Prove that the expression*

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Problem 23 (00B5). *Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.*

Problem 24 (99A6). *The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,*

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

Problem 25 (99B6). *Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n) = 1$ or $\gcd(s,n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s,t)$ is prime.*

Problem 26 (98A4). *Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.*

Problem 27 (98B4). *Find necessary and sufficient conditions on positive integers m and n so that*

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

Problem 28 (98B5). *Let N be the positive integer with 1998 decimal digits, all of them 1; that is,*

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

Problem 29 (98B6). *Prove that, for any integers a, b, c, there exists a positive integer n such that*

$$\sqrt{n^3 + an^2 + bn + c}$$

is not an integer.

Problem 30 (97A5). *Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \cdots, a_n)$ such that*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1.$$

Determine whether N_{10} is even or odd.

Problem 31 (97A6). *For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,*

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

Problem 32 (97B1). *For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,*

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

Problem 33 (97B3). *For each positive integer n, write the sum*

$$\sum_{m=1}^{n} \frac{1}{m}$$

in the form $\frac{p_n}{q_n}$, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

Problem 34 (97B5). *Prove that for $n \geq 2$,*

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

Problem 35 (96A5). *If p is a prime number greater than 3 and $k = \lfloor \frac{2p}{3} \rfloor$, prove that the sum*

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

Problem 36 (96B1). *Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \cdots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.*

Problem 37 (95A3). *The number $d_1 d_2 \cdots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1 d_2 \ldots d_9$ by the corresponding digit e_i ($1 \leq i \leq 9$) is divisible by 7. The number $f_1 f_2 \ldots f_9$ is related to $e_1 e_2 \ldots e_9$ is the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 \ldots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]*

Problem 38 (95A4). *Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy*

$$\sum_{i=1}^{k} x_i \leq k - 1 \qquad \textit{for} \quad k = 1, 2, \ldots, n.$$

Problem 39 (95B6). *For a positive real number α, define*

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}.$$

Prove that $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$. [As usual, $\lfloor x \rfloor$ is the greatest integer $\leq x$.]

Problem 40 (94A4). *Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.*

Problem 41 (94B1). *Find all positive integers n that are within 250 of exactly 15 perfect squares.*

Problem 42 (94B5). *For any real number α, define the function $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,*

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

Problem 43 (94B6). *For any integer n, set*

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$.

Problem 44 (93A4). *Let $x_1, x_2, \dots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \dots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.*

Problem 45 (93A6). *The infinite sequence of 2's and 3's*

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3,$$
$$3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \dots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Problem 46 (93B1). *Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which*

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Problem 47 (93B5). *Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.*

Problem 48 (93B6). *Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x < y$ and replace them with $2x$ and $y - x$.*

Problem 49 (92A1). *Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.*

(i) $f(f(n)) = n$, *for all integers n;*

(ii) $f(f(n+2)+2) = n$ *for all integers n;*

(iii) $f(0) = 1$.

Problem 50 (92A3). *For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy*

$$(x^2 + y^2)^m = (xy)^n.$$

Problem 51 (92A5). *For each positive integer n, let $a_n = 0$ (or 1) if the number of 1's in the binary representation of n is even (or odd), respectively. Show that there do not exist positive integers k and m such that*

$$a_{k+j} = a_{k+m+j} = a_{k+m+2j},$$

for $0 \leq j \leq m - 1$.

Problem 52 (91A61). *Let $A(n)$ denote the number of sums of positive integers*

$$a_1 + a_2 + \cdots + a_r$$

which add up to n with

$$a_1 > a_2 + a_3, a_2 > a_3 + a_4, \cdots, a_{r-2} > a_{r-1} + a_r, a_{r-1} > a_r.$$

Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

1. $b_1 \geq b_2 \geq \cdots \geq b_s$,
2. *each* b_i *is in the sequence* $1, 2, 4, \ldots, g_j, \ldots$ *defined by* $g_1 = 1$, $g_2 = 2$, *and* $g_j = g_{j-1} + g_{j-2} + 1$, *and*
3. *if* $b_1 = g_k$ *then every element in* $\{1, 2, 4, \ldots, g_k\}$ *appears at least once as a* b_i.

Prove that $A(n) = B(n)$ *for each* $n \geq 1$. *(For example,* $A(7) = 5$ *because the relevant sums are* $7, 6+1, 5+2, 4+3, 4+2+1$, *and* $B(7) = 5$ *because the relevant sums are* $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.*)*

Problem 53 (91B4). *Suppose* p *is an odd prime. Prove that*

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

Problem 54 (91B5). *Let* p *be an odd prime and let* Z_p *denote (the field of) integers modulo* p. *How many elements are in the set*

$$\{x^2 : x \in Z_p\} \cap \{y^2 + 1 : y \in Z_p\}?$$

Problem 55 (90A3). *Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to* $\frac{5}{2}$.

Problem 56 (90A4). *Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?*

Problem 57 (89B1). *How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?*

Problem 58 (88B1). *A* composite *(positive integer) is a product* ab *with* a *and* b *not necessarily distinct integers in* $\{2, 3, 4, \cdots\}$. *Show that every composite is expressible as* $xy + xz + yz + 1$, *with* x, y, z *positive integers.*

Problem 59 (88B3). *For every* n *in the set* $\mathrm{N} = \{1, 2, \ldots\}$ *of positive integers, let* r_n *be the minimum value of* $|c - d\sqrt{3}|$ *for all nonnegative integers* c *and* d *with* $c + d = n$. *Find, with proof, the smallest positive real number* g *with* $r_n \leq g$ *for all* $n \in \mathrm{N}$.

Problem 60 (88B6). *Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$.)*

Problem 61 (87A2). *The sequence of digits*

$$123456789101112131415161718192021\dots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Problem 62 (86A2). *What is the units (i.e., rightmost) digit of*

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Problem 63 (85A4). *Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?*

Problem 64 (85B1). *Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial*

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

Problem 65 (85B3). *Let*

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

Problem 66 (84A6). *Let n be a positive integer, and let $f(n)$ denote the last nonzero digit in the decimal expansion of $n!$. For instance, $f(5) = 2$.*

(a) Show that if $a_1, a_2, \cdots, a_k$ are distinct nonnegative integers, then $f\left(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}\right)$ depends only on the sum $a_1 + a_2 + \cdots + a_k$.

(b) Assuming part (a), we can define $g(s) = f\left(5^{a_1} + 5^{a_2} + + 5^{a_k}\right)$, where $s = a_1 + a_2 + \cdots + a_k$. Find the least positive integer p for which $g(s) = g(s+p)$, for all $s \geq 1$, or else show that no such p exists.

Problem 67 (84B1). *Let n be a positive integer, and define*

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

Problem 68 (84B5). *For each nonnegative integer k, let $d(k)$ denote the number of 1s in the binary expansion of k (for example, $d(0) = 0$ and $d(5) = 2$). Let m be a positive integer. Express*

$$\sum_{k=0}^{2^m-1} (-1)^{d(k)} k^m$$

in the form $(-1)^m a^{f(m)}(g(m))!$, where a is an integer and f and g are polynomials.

Problem 69 (83A1). *How many positive integers n are there such that n is an exact divisor of at least one of the numbers*

$$10^{40},\ 20^{30}?$$

Problem 70 (83A3). *Let p be in the set $\{3, 5, 7, 11, \cdots\}$ of odd primes and let*

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \cdots, p-1\}$, then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

Problem 71 (83B2). *For positive integers n, let $C(n)$ be the number of representations of n as a sum of non -increasing powers of 2, where no power can be used more than three times. For example $C(8) = 5$ since the representations for 8 are:*

$$8, 4+4, 4+2+2, 4+2+1+1, 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n; here $[u]$ denotes the greatest integer less than or equal to u.

Problem 72 (83B4). *Let $f(n) = n + [\sqrt{n}]$ where $[x]$ is the largest integer less than or equal to x. Prove that, for every positive integer m, the sequence*

$$m, f(m), f(f(m)), f(f(f(m))), \cdots$$

contains at least one square of an integer.

Problem 73 (82A5). *Let a, b, c, and d be positive integers and*

$$r = 1 - \frac{a}{b} - \frac{c}{d}$$

Given that $a + c < 1982$ and $r > 0$, prove that

$$r > \frac{1}{1983^3}.$$

Problem 74 (82B4). *Let $n_1, n_2, \cdots, n_s$ be distinct integers such that*

$$(n_1 + k)(n_1 + k) \cdots (n_s + k)$$

is an integral multiple of $n_1 \cdots n_s$ for every integer k. For each of the following assertions, give a proof or a counterexample:

(a) $|n_i| = 1$ for some i.

(b) If further all n_i are positive, then $(n_1, n_2, \cdots, n_s) = (1, 2, \cdots, s)$.

Problem 75 (81B3). *Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of $n^2 + 3$, then p is also a divisor of $k^2 + 3$ for some integer k with $k^2 < n$.*

Problem 76 (80A2). *Let r and s be positive integers. Derive a formula for the number of ordered quadruples (a, b, c, d) of positive integers such that*

$$3^r 7^s = lcm(a, b, c) = lcm(a, b, d) = lcm(a, c, d) = lcm(b, c, d).$$

The answer should be a function of r and s. (Note that $lcm(x, y, z)$ denotes the least common multiple of x, y, z.)

Problem 77 (80A4). *(a) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that*

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$$

(b) Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$$

Problem 78 (80B3). *For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in the simplest form.)*

Problem 79 (80B6). *An infinite array of rational numbers $G(d, n)$ is defined for integers d and n with $1 \leq d \leq n$ as follows:*

$$G(1, n) = \frac{1}{n},\ G(d, n) = \frac{d}{n} \sum_{i=d}^{n} G(d-1, i-1),\ d > 1.$$

For $1 < d < p$ and p prime, prove that $G(d, p)$ is expressible as a quotient $\frac{s}{t}$ of integers s and t with t not an integral multiple of p. (For example, $G(3, 5) = 7 = 4$ with the denominator 4 not a multiple of 5.)