**1** (Putnam 1991/B4) Suppose that $p$ is an odd prime. Prove that

PEN D2

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

*First Solution.* (90% number theory + 10% combinatorics) We first offer three well-known properties on binomial coefficients.

**Lemma 1.** *Let $p$ be a prime and let $k \in \{1, \cdots, p-1\}$. Then, we have*

$$\begin{cases} (a) & \binom{p}{k} \equiv 0 \ (mod\ p), \\ (b) & \binom{p+k}{k} \equiv 1 \ (mod\ p), \\ (c) & \binom{2p}{p} \equiv 2 \ \left(mod\ p^2\right). \end{cases}$$

Proof of Lemma 1 For (a) and (b), we work on the field $\mathbb{Z}/p\mathbb{Z}$ and identify the coset $\overline{a} = a + p\mathbb{Z}$ with $a \in \mathbb{Z}$. We compute

$$(a)\ \ \binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} = \frac{0}{k}\binom{p-1}{k-1} = 0,$$

and

$$(b)\ \ \binom{p+k}{k} = \frac{(p+k)!}{k!p!} = \frac{1}{k!}\prod_{i=1}^{k}(p+i) = \frac{1}{k!}\prod_{i=1}^{k} i = 1.$$

It follows from Vandermonde's Identity and (a) that

$$(c)\ \ \binom{2p}{p} \equiv \sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k} \equiv 1 + \sum_{k=1}^{p-1}\binom{p}{k}^2 + 1 \equiv 2 \left(mod\ p^2\right).$$

Now, we prove the congruence in the problem. By (a) and (b) in Lemma 7, whenever $j \in \{1, \cdots, p-1\}$, the integer $\left(\binom{p+j}{j} - 1\right)\binom{p}{j}$ is divisible by $p^2$, in other words, $\binom{p}{j}\binom{p+j}{j} \equiv \binom{p}{j} \ \left(mod\ p^2\right)$.

It follows from this, Lemma 1 and Binomial Theorem that

$$\begin{aligned} \sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} &\equiv 1 + \left(\sum_{j=1}^{p-1}\binom{p}{j}\binom{p+j}{j}\right) + \binom{2p}{p} \pmod{p^2} \\ &\equiv 1 + \sum_{j=1}^{p-1}\binom{p}{j} + 2 \pmod{p^2} \\ &\equiv 1 + (2^p - 2) + 2 \pmod{p^2} \\ &\equiv 2^p + 1 \pmod{p^2}. \end{aligned}$$

□

*Second Solution.* (1% number theory + 99% combinatorics) We establish the following combinatorial identity.

**Lemma 2.** *For all positive integers $n$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}\binom{n}{k}^2 2^k.$$

PROOF OF LEMMA 2 We first expand the polynomial $(2+x)^n(1+x)^n = ((1+x)+1)^n(1+x)^n$ in two ways. On the one hand, we compute

$$f(x)=\left(\sum_{k=0}^{n}\binom{n}{k}2^k x^{n-k}\right)\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)=\sum_{l=0}^{2n}\left(\sum_{k+j=l,0\le k,j\le n}\binom{n}{k}\binom{n}{j}2^k\right)x^l.$$

On the other hand, we compute

$$\begin{aligned} f(x) &= \left(\sum_{k=0}^{n}\binom{n}{k}(1+x)^k\right)(1+x)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}(1+x)^{n+k} \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\sum_{j=0}^{n+k}\binom{n+k}{j}x^j\right) \\ &= \sum_{j=0}^{2n}\left(\sum_{k=max(0,n-j)}^{n}\binom{n}{k}\binom{n+k}{j}\right)x^j \end{aligned}$$

Now, we can find the coefficient of $x^n$ in $f(x)$ in two ways. The first identity gives

$$x^n[f(x)]=\sum_{k+j=n}\binom{n}{k}\binom{n}{j}2^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}2^k=\sum_{k=0}^{n}\binom{n}{k}^2 2^k$$

and the second identity gives

$$x^n[f(x)]=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}.$$

Equating the coefficients $x^n[f(x)]$, we get the desired result.

Now, we go back to the original problem. We take $n=p$ in LEMMA 2 and use the fact that, $\binom{p}{k}$ is divisible by $p$, where $k\in\{1,\cdots,p-1\}$. We obtain

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 1+\sum_{j=1}^{p-1}\binom{p}{j}^2 2^j + 2^p \equiv 1+2^p \pmod{p^2}.$$

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