1 ((D. Fomin) [Ams, pp. 12]) Consider the set of all five-digit numbers whose decimal
PEN O49 representation is a permutation of the digits $1, 2, 3, 4, 5$.
Prove that this set can be divided into two groups, in such a way that the sum of the squares of the numbers in each group is the same.

Proof. For this problem, let us denote

$$\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\} : i \mapsto 6 - i$$

and

$$S = \{n \in \mathbb{N} | n > 33333 \text{ and } n \text{ is a digit permutation of } 12345\}.$$

Also, we will write $[a, b, c, d, e]$ for the number $10000a + 1000b + 100c + 10d + e$, and we define

$$[a, b, c, d, e]^\sigma := [\sigma(a), \sigma(b), \sigma(c), \sigma(d), \sigma(e)].$$

Now, we can represent each number in $s \in S$ as $s = [a, b, c, d, e]$, with $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$. We define the difference T_{abcde} as

$$T_{abcde} = [a, b, c, d, e]^2 - [\sigma(a), \sigma(b), \sigma(c), \sigma(d), \sigma(e)]^2,$$

which rewrites as $[6, 6, 6, 6, 6] \cdot [a - \sigma(a), b - \sigma(b), c - \sigma(c), d - \sigma(d), e - \sigma(d)]$. Also, observe that when $[a, b, c, d, e] > 33333$, then we automatically have $[\sigma(a), \sigma(b), \sigma(c), \sigma(d), \sigma(e)] < 33333$ and vice versa (since the sum is 66666), hence exactly half of our five-digit numbers are in S (and σ is a bijection between the elemens in S and the elements not in S).

Now, consider the following problem: let T be the set

$$T := \{T_{abcde} | [a, b, c, d, e] \in S\}.$$

Prove that we can split T into two disjoint sets T_1, T_2 with equal sum of elements.

If we find such T_1, T_2, then

$$S_1 := \{[a, b, c, d, e] | [a, b, c, d, e] \in T_1\} \cup \{[a, b, c, d, e]^\sigma | [a, b, c, d, e] \in T_2\}$$

$$S_2 := \{[a, b, c, d, e] | [a, b, c, d, e] \in T_2\} \cup \{[a, b, c, d, e]^\sigma | [a, b, c, d, e] \in T_1\}$$

is a valid partition for our original problem.

That leaves us to find T_1, T_2. First observe that, if we define

$$\tau : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\} : (12)(3)(4)(5).$$

then $[a, b, c, d, e] > 33333$ if and only if $[a, b, c, d, e]^\tau := [\tau(a), \tau(b), \tau(c), \tau(d), \tau(e)] > 33333$, hence we can split the elements of T into pairs (t_1, t_2) with $t_1 > t_2$, where the only difference is the switched position of 1 and 2.

For $[a, b, c, d, e] > 33333$, we need $a \geq 3$, so the difference $t_1 - t_2 \in \{9, 90, 99, 900, 990, 999\}$, appearing respectively $3!, 3!, 3!, 3! - 2!, 3! - 2!, 3! - 2!$ times. To find our T_1, T_2, it is sufficient to partition the set of couples (t_1, t_2) into two sets A and B, such that

$$\sum_{(t_1,t_2)\in T_1} t_1 - t_2 = \sum_{(t_1,t_2)\in T_2} t_1 - t_2,$$

since this will cause them to have equal sum of elements.

Now note that

$$999 = 990 + 9,$$

$$999 = 900 + 99,$$

$$990 = 900 + 90$$

and

$$99 = 90 + 9.$$

Hence:

1. Put 3 couples with $t_1 - t_2 = 999$ into A, 3 couples with $t_1 - t_2 = 990$ into B, 3 couples with $t_1 - t_2 = 9$ into B.

2. Put 3 couples with $t_1 - t_2 = 999$ into A, 3 couples with $t_1 - t_2 = 900$ into B, 3 couples with $t_1 - t_2 = 99$ into B.

3. Put 3 couples with $t_1 - t_2 = 990$ into A, 3 couples with $t_1 - t_2 = 900$ into B, 3 couples with $t_1 - t_2 = 90$ into B.

4. Put 1 couple with $t_1 - t_2 = 99$ into A, 1 couple with $t_1 - t_2 = 90$ into B, 1 couple with $t_1 - t_2 = 9$ into B.

By the counting above, we have partitioned all couples into two sets, which have equal sum of elements per construction.

Now, we define

$$T_1 := \{t_1 | (t_1, t_2) \in A\} \cup \{t_2 | (t_1, t_2) \in B\}$$

and

$$T_2 := \{t_2 | (t_1, t_2) \in A\} \cup \{t_1 | (t_1, t_2) \in B\}.$$

Since A, B was a partition of the couples of elements of S, this is a valid partition. Quod erat demonstrandum. $\square$