PEN Problems Book Extended

Version 0.9 (25th June, 2008)

OVERTURE

Welcome to PEN Problems Book[Extended]! The main purpose of Project PEN is to share an extensive collection of challenging problems with students, teachers and problem creators. This is a joint-work with Project PEN Team and The IMO Compendium Group (http://www.imo.org.yu). We owe great debts to Dusan Djukic for providing us with TeX files.

PEN TEAM

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Supporting Sites

PEN Website http://www.problem-solving.be/pen
MATHLINKS http://www.mathlinks.ro/index.php?f=456
PEN Blog http://projectpen.wordpress.com

How to Help Us

Currently, we are working hard on this *forever* project. WE NEED STAFFS. If you wish to be a new member of PEN TEAM, then please email us!

Contents

Chapter 1

Year 2007

1 Let x, y be integers different from -1 such that

$$\frac{x^4-1}{y+1}+\frac{y^4-1}{x+1}$$

is an integer. Prove that $x^4y^{44}-1$ is divisible by $x+1$.

Vietnam 2007

2 Find a solution to the equation $x^2-2x-2007y^2=0$ in positive integers.

NrdMO 2007

3 For each positive number x define $A(x)=\{[nx] \mid n \in \mathbb{N}\}$. Find all irrational numbers $\alpha > 1$ with the following property: If a positive number β is such that $A(\alpha) \supset A(\beta)$, then β/α is an integer.

Japan 2007

4 We say that polynomials p and q with integer coefficients are *similar* if they have the same degree and the coefficients which differ only in order.

1. Prove that if p and q are similar then $p(2007)-q(2007)$ is even.
2. Is there an integer $k > 2$ such that $p(2007)-q(2007)$ is a multiple of k for any two similar polynomials p and q?

Italy 2007

5 Consider the sequence given by $x_1=2$, $x_{n+1}=2x_n^2-1$ for $n \geq 1$. Prove that n and x_n are coprime for each $n \geq 1$.

Italy 2007

6 Let $p \geq 5$ be a prime number.

1. Show that there exists a prime divisor $q \neq p$ of $N = (p-1)^p + 1$.
2. If $\prod_{i=1}^n p_i^{a_i}$ is the canonical factorization of N, prove that
$$\sum_{i=1}^n a_i p_i \geq \frac{p^2}{2}.$$

ITALY 2007

7 Find all prime numbers p and q such that p divides $q+6$ and q divides $p+7$.

IRELAND 2007

8 Let r and n be nonnegative integers such that $r \leq n$.

1. Prove that
$$\frac{n+1-2r}{n+1-r}\binom{n}{r}$$
is an integer.
2. Prove that
$$\sum_{r=0}^{[n/2]} \frac{n+1-2r}{n+1-r}\binom{n}{r} < 2^{n-2}$$
for all $n \geq 9$.

IRELAND 2007

9 Find the number of zeros in which the decimal expansion of 2007! ends. Also find its last non-zero digit.

IRELAND 2007

10 Find all nonnegative integers $a < 2007$ for which the congruence $x^2 + a \equiv 0 \pmod{2007}$ has exactly two different nonnegative integer solutions smaller than 2007.

AUSTRIA 2007

11 Solve in nonnegative integers $x_1, \ldots, x_6$ the system of equations
$$x_k x_{k+1}(1 - x_{k+2}) = x_{k+3}x_{k+4}, \quad k = 1, \ldots, 6,$$
where $x_{k+6} = x_k$.

AUSTRIA 2007

12 Find all triples of positive prime numbers p_1, p_2, p_3 such that

$$\begin{cases} 2p_1 - p_2 + 7p_3 & = 1826, \\ 3p_1 + 5p_2 + 7p_3 & = 2007. \end{cases}$$

Belarus 2007

13 Prove that the number

$$\sqrt{8.\underbrace{000\ldots01}_{n}}$$

is irrational for every positive integer n.

Belarus 2007

14 Find all positive integers k with the following property: There are four distinct divisors k_1, k_2, k_3, k_4 of k such that k divides $k_1 + k_2 + k_3 + k_4$.

Belarus 2007

15 Find all positive integers n and m satisfying the equality

$$n^5 + n^4 = 7^m - 1.$$

Belarus 2007

16 Let (m, n) be a pair of positive integers.

1. Prove that the set of all positive integers can be partitioned into four pairwise disjoint nonempty subsets such that none of them has two numbers with absolute value of their difference equal to either m, n, or $m + n$.

2. Find all pairs (m, n) such that the set of all positive integers can not be partitioned into three pairwise disjoint nonempty subsets satisfying the above condition.

Belarus 2007

17 The set M contains all natural numbers from 1 to 2007 inclusive and has the following property: If $n \in M$, then M contains all terms of the arithmetic progression with first term n and difference $n + 1$. Decide whether there must always exist a number m such that M contains all natural numbers greater than m.

CSMO[a] 2007

[a]Czech and Slovak Mathematical Olympiad

18 Given a positive integer a, how many nonnegative integer solutions does the equation $\left[\frac{x}{a}\right] = \left[\frac{x}{a+1}\right]$ have?

Germany 2007

19 If n is is an integer such that $4n+3$ is divisible by 11, find the from of n and the remainder of n^4 upon division by 11.

Greece 2007

20 For an integer n, denote $A = \sqrt{n^2+24}$ and $B = \sqrt{n^2-9}$. Find all values of n for which $A - B$ is an integer.

Greece 2007

21 Find all natural numbers n for which the number $2007 + n^4$ is a perfect square.

Greece 2007

22 Let a, b, c be natural numbers and $a^2+b^2+c^2 = n$. Prove that there exist constants p_i, q_i, r_i $(i = 1, 2, 3)$ independent of a, b, c such that

$$(p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_2a + q_2b + r_2c)^2 = 9n.$$

Further, if a, b, c are not all divisible by 3, show that $9n$ can be expressed as $x^2+y^2+z^2$ for some natural numbers x, y, z not divisible by 3.

India 2007

23 The equation $x^2 - mx + n = 0$ has real roots α and β, where m and n are positive integers. Prove that α and β are integers if and only if $[m\alpha]+[m\beta]$ is a perfect square.

India 2007

24 Let $P(x)$ and $Q(x)$ be the polynomials with integer coefficients. If $P(x)$ is monic, prove that there exists a monic polynomial $R(x) \in \mathbb{Z}[x]$ such that

$$P(x) \mid Q(R(x)).$$

Iran 2007

25 Let A be the largest subset of $\{1, \ldots, n\}$ such that each element of A divides the most one other element of A. Prove that

$$\frac{2n}{3} \leq |A| \leq 3\left\lceil \frac{n}{4} \right\rceil.$$

Iran 2007

26 Does there exist a sequence of positive integers $a_0, a_1, a_2, \ldots$ such that for each $i \neq j$, $(a_i, a_j) = 1$ and for all n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

Iran 2007

27 Natural numbers a, b and c are pairwise distinct and satisfy

$$a \mid b + c + bc, \quad b \mid c + a + ca, \quad c \mid a + b + ab.$$

Prove that at least one of the numbers a, b, c is not prime.

Macedonia 2007

28 Find the number of subsets of the set $\{1, 2, 3, \ldots, 5n\}$ such that the sum of the elements in each subset are divisible by 5.

Mongolia 2007

29 If $x, y, z \in \mathbb{N}$ and $xy = z^2 + 1$ prove that there exist integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$, $z = ac + bd$.

Mongolia 2007

30 Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \geq 2$. If for any $\alpha \in \mathbb{N}$, $p_i - 1 \nmid \alpha$, where $i = 1, 2, \ldots, s$, prove that

$$n \mid \sum_{\alpha \in \mathbb{Z}_n^*} a^{\alpha}$$

where $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n : (a, n) = 1\}$.

Mongolia 2007

31 Natural numbers from 1 to 100 are arranged in a 10×10 board. In each step it is allowed to exchange places of two numbers. Show that one can always perform 35 steps so that in the resulting board the sum of any two numbers adjacent by side is a composite number.

Russia 2007

32 For an integer $n > 3$ denote by $n?$ the product of all prime numbers less than n. Solve the equation $n? = 2n + 16$.

Russia 2007

33 Do there exist nonzero numbers a, b, c such that for each $n > 3$ there is a polynomial of the form $P_n(x) = x^n + \cdots + ax^2 + bx + c$ having exactly n integral roots (not necessarily distinct)?

Russia 2007

34 Determine all pairs of natural numbers (x, n) that satisfy the equation

$$x^3 + 2x + 1 = 2^n.$$

Serbia 2007

35 Determine all natural numbers n for which there exists a permutation σ of numbers $1, 2, \ldots, n$ such that the number

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}$$

is rational.

BMO[a] 2007

[a]Balkan Mathematical Olympiads

Chapter 2

Year 2006

1 Determine the largest natural number whose all decimal digits are different and which is divisible by each of its digits.

Serbia and Montenegro 2006

2 For every natural number a, consider the set $S(a) = \{a^n + a + 1 \mid n = 2, 3, \dots\}$. Does there exist an infinite set $A \subset \mathbb{N}$ with the property that for any two distinct elements $x, y \in A$, x and y are coprime and $S(x) \cap S(y) = \emptyset$?

Serbia and Montenegro 2006

3 Given prime numbers p and q with $p < q$, determine all pairs (x, y) of positive integers such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{p} - \frac{1}{q} .$$

Serbia and Montenegro 2006

4 A set T is called *naughty* if for any two (not necessarily distinct) elements u, v of T, $u + v \notin T$. Prove that

1. a naughty subset of $S = \{1, 2, \dots, 2006\}$ has at most 1003 elements;
2. every set S of 2006 positive numbers contains a naughty subset having 669 elements.

Vietnam 2006

5 A sequence (a_n) of positive integers is given by $a_0 = m$ and

$$a_{n+1} = a_n^5 + 487 \quad \text{for } n \geq 0.$$

Determine all values of m for which this sequence contains the maximum possible number of squares.

NrdMO 2006

6 Find all integers k for which there exist infinitely many triples (a, b, c) of integers satisfying $(a^2 - k)(b^2 - k) = c^2 - k$.

Japan 2006

7 Find all triples (m, n, p) such that $p^n + 144 = m^2$, where m and n are positive integers and p a prime number.

Italy 2006

8 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers m, n

$$f(m - n + f(n)) = f(m) + f(n).$$

Italy 2006

9 For each positive integer n, let A_n denote the set of positive integers $a \leq n$ such that $n \mid a^n + 1$.

1. Find all n for which A_n is nonempty.
2. Find all n for which $|A_n|$ is even and nonzero.
3. Is there an n with $|A_n| = 130$?

Italy 2006

10 If natural numbers x, y, p, n, k with $n > 1$ odd and p an odd prime satisfy $x^n + y^n = p^k$, prove that n is a power of p.

HIBMC 2006

11 Consider the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 2$. Find the number of subsets B of A such that for any two elements of A whose sum is a power of 2 exactly one of them is in B.

Bulgaria 2006

12 Let p be a prime number such that p^2 divides $2^{p-1}-1$. Show that for any natural n the number $(p-1)(p!+2^n)$ has at least three distinct prime divisors.

Bulgaria 2006

13 A positive integer is called *bold* if it has 8 positive divisors that sum up to 3240. For example, 2006 is bold because its divisors $1, 2, 17, 34, 59, 118, 1003, 2006$ have the sum 3240. Find the smallest bold number.

Brazil 2006

14 Determine all triples (m, n, p) of positive rational numbers such that the numbers

$$m + \frac{1}{np}, \quad n + \frac{1}{pm}, \quad p + \frac{1}{mn}$$

are integers.

BMO 2006

15 Let m be a positive integer. Find all positive integers a such that the sequence $(a_n)_{n=0}^{\infty}$ defined by $a_0 = a$ and

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even,} \\ a_n + m & \text{if } a_n \text{ is odd} \end{cases} \quad \text{for } n = 0, 1, 2, \ldots$$

is periodic (there exists $d > 0$ such that $a_{n+d} = a_n$ for all n).

BMO 2006

16 For each permutation (a, b, c, d, e, f) of the set $M(n) = \{n, n+1, \ldots, n+6\}$, where n is a positive integer, consider the sum $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$. Let

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = \frac{xvw + yuw + zuv}{uvw}$$

be the greatest of these sums.

1. Prove that if n is odd then $\gcd(xvw + yuw + zuv, uvw) = 1$ if and only if $\gcd(x, u) = \gcd(y, v) = \gcd(z, w) = 1$.
2. For which n does $\gcd(xvw + yuw + zuv, uvw) = 1$ hold?

APMC 2006

17 A positive integer d is called *nice* if for any positive integers x, y the number $(x+y)^5 - x^5 - y^5$ is a multiple of d if and only if so is $(x+y)^7 - x^7 - y^7$.

1. Is 29 a nice number?
2. Is 2006 a nice number?
3. Show that there are infinitely many nice numbers.

APMC 2006

18 A natural number n ends with exactly k zeros in decimal representation and is greater than 10^k. Find, as a function of k, the smallest possible number of representations of n as a difference of two perfect squares.

Austria 2006

19 For each positive number x define $f(x) = [x^2] + \{x\}$ (where $[u]$ is the integral and $\{u\}$ the fractional part of u). Show that there exists a nonconstant arithmetic sequence of positive rational numbers which all have the denominator 3 in the reduced form and none of which occurs as a value of f.

Austria 2006

20 Let N be a positive integer. Find the number of natural numbers $n \leq N$ which have a multiple whose decimal representation consists of digits 2 and 6 only.

Austria 2006

21 For which rational x is $1 + 105 \cdot 2^x$ a square of a rational number?

Austria 2006

22 Let A be a nonzero integer. Solve the following system in integers:

$$\begin{aligned} x + y^2 + z^3 &= A \\ \tfrac{1}{x} + \tfrac{1}{y^2} + \tfrac{1}{z^3} &= \tfrac{1}{A} \\ xy^2z^3 &= A^2. \end{aligned}$$

Austria 2006

23 *Lagrange's Theorem:* Every positive integer can be written as a sum of four perfect squares.
Find all natural numbers that can be uniquely expressed as a sum of at most five perfect squares.

Croatia 2006

24 Find all solutions of the equation $3^x = 2^x y + 1$ in positive integers.

Croatia 2006

25 The sequence $(a_n)_{n=1}^{\infty}$ of positive integers satisfies $a_{n+1} = a_n + b_n$ for each $n \geq 1$, where b_n is obtained from a_n by reversing its digits (number b_n may start with zeros). For instance, $a_1 = 170$ yields $a_2 = 241$, $a_3 = 383$, $a_4 = 766$, etc. Decide whether a_7 can be a prime number.

CSMO 2006

26 Let m and n be natural numbers for which the equation

$$(x+m)(x+n) = x+m+n$$

has at least one integer solution. Show that $\frac{1}{2} < \frac{m}{n} < 2$.

CSMO 2006

27 Suppose that a, b are positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all $n \in \mathbb{N}$. Prove that $a = b$.

CSMO 2006

28 Suppose that a, b are positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all $n \in \mathbb{N}$. Prove that $a = b$.

France 2006

29 The set $M = \{1, 2, \ldots, 3n\}$ is partitioned into three subsets A, B, C of cardinality n. Show that there exist numbers a, b, c in three different subsets such that $a = b + c$.

France 2006

30 Find two consecutive natural numbers each of which has the sum of digits divisible by 2006.

Germany 2006

31 Prove that the equation $x^3 + y^3 = 4(x^2 y + xy^2 + 1)$ has no integer solutions.

Germany 2006

32 Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}$ that satisfy the equality

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)} \quad \text{for all } x, y \in \mathbb{Q}^+.$$

GERMANY 2006

33 A positive integer is called *digit-reduced* if at most nine different digits occur in its decimal representation. (Leading zeros are omitted.) Let M be a finite set of digit-reduced integers. Prove that the sum of the reciprocals of the elements of M is less than 180.

GERMANY 2006

34 Find all pairs of positive integers (x, y) such that $2x^y - y = 2005$.

GREECE 2006

35 Prove that among any 27 distinct positive integers less than 100 there exist two that are not coprime.

GREECE 2006

36 Prove that if n is a positive integer, then the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

has no solution in rational numbers x, y.

GREECE 2006

37 Find all pairs (a, b) of positive integers such that $2a - 1$ and $2b + 1$ are coprime and $a + b$ divides $4ab + 1$.

IBMO[a] 2006

[a]Iberoamerican Mathematical Olympiad

38 Prove that for every positive integer n there is a unique ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a + b - 1)(a + b - 2) + a.$$

INDIA 2006

39 Prove that for each $n \geq 4011^2$ there is an integer l with $n < l^2 < \left(1 + \frac{1}{2005}\right) n$. Find the smallest M such that, for each integer $n \geq M$, there is an integer l with $n < l^2 < \left(1 + \frac{1}{2005}\right) n$.

INDIA 2006

40 For a positive integer a, let S_a be the set of primes p for which there exists an odd integer b such that p divides $\left(2^{2^a}\right)^b - 1$. Prove that for every a there exist infinitely many primes that are not contained in S_a.

KOREA 2006

41 Determine all nonnegative integers n for which $2^n + 105$ is a perfect square.

POLAND 2006

42 A prime number $p > 3$ and positive integers a, b, c satisfy $a + b + c = p + 1$ and the number $a^3 + b^3 + c^3 - 1$ is divisible by p. Show that at least one of the numbers a, b, c is equal to 1.

POLAND 2006

43 Let $k_1 < k_2 < \cdots < k_m$ be nonnegative integers. Define $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_m}$. Find the number of odd coefficients of the polynomial $P(x) = (x + 1)^n$.

POLAND 2006

44 Positive integers a, b, c and x, y, z satisfy $|x - a| \leq 1$, $|y - b| \leq 1$, and

$$a^2 + b^2 = c^2, \quad x^2 + y^2 = z^2.$$

Prove that the sets $\{a, b\}$ and $\{x, y\}$ coincide.

POLAND 2006

45 Given a natural number c, we define the sequence (a_n) by $a_1 = 1$ and

$$a_{n+1} = d(a_n) + c \quad \text{for } n = 1, 2, \ldots,$$

where $d(m)$ denotes the number of positive divisors of $m \in \mathbb{N}$. Show that there is a positive integer k such that the sequence $a_k, a_{k+1}, a_{k+2}, \ldots$ is periodic.

POLAND 2006

46 A prime number p and an integer n with $p \geq n \geq 3$ are given. A set A of sequences of length n with terms in the set $\{0, 1, 2, \ldots, p-1\}$ has the following property: Any two sequences $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ from A differ in at least three positions. Find the largest possible cardinality of A.

Poland 2006

47 Find all positive integers k for which the number $3^k + 5^k$ is a power of an integer with the exponent greater than 1.

Poland 2006

48 Find all pairs of integers (a, b) for which there exists a polynomial $P(x)$ with integer coefficients such that the product $(x^2 + ax + b)P(x)$ is a polynomial of the form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

where each of $c_0, \ldots, c_{n-1}$ is equal to 1 or -1.

Poland 2006

49 Let A be a set of at least two positive integers. Suppose that for each $a, b \in A$ with $a > b$ we have $\frac{[a,b]}{a-b} \in A$. Show that set A has exactly two elements.

Romania 2006

50 Let n be a positive integer. Show that there exist an integer $k \geq 2$ and numbers $a_1, a_2, \ldots, a_k \in \{-1, 1\}$ such that

$$n = \sum_{1 \leq i < j \leq k} a_i a_j.$$

Romania 2006

51 Show that the sequence given by $a_n = \left[n\sqrt{2}\right] + \left[n\sqrt{3}\right]$, $n = 0, 1, \ldots$ contains infinitely many even numbers and infinitely many odd numbers.

Romania 2006

52 Let K be a finite field. Prove that the following statements are equivalent:

1. $1 + 1 = 0$;
2. For each $f \in K[x]$ with $\deg f \geq 1$, $f(x^2)$ is reducible.

Romania 2006

53 how that there exist four integers a, b, c, d whose absolute values are greater than $1,000,000$ such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{abcd}.$$

Russia 2006

54 Let $a_1 < a_2 < \cdots < a_{10}$ be natural numbers and let b_k be the largest divisor of a_k with $b_k < a_k$. Suppose that $b_1 > b_2 > \cdots > b_{10}$. Prove that $a_{10} > 500$.

Russia 2006

55 Suppose that the sum of cubes of three consecutive positive integers is a perfect cube. Prove that among the three integers, the middle one is divisible by 4.

Russia 2006

56 The sum and product of two purely periodic decimal numbers is is a purely periodic decimal number of period T. Prove that the two initial numbers have periods not exceeding T.

Russia 2006

57 Suppose that the polynomial $(x+1)^n - 1$ is divisible by a polynomial $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0$ of an even degree k whose all coefficients $c_0, \ldots, c_{k-1}$ are odd integers. Prove that n is divisible by $k+1$.

Russia 2006

58 If positive integers a and b have 99 and 101 different positive divisors respectively (including 1 and the number itself), can the product ab have exactly 150 positive divisors?

Sweden 2006

59 Determine all positive integers a, b, c satisfying $a^{(b^c)} = (b^a)^c$.

Sweden 2006

60 A subset M of $\{1, 2, \ldots, 2006\}$ has the property that for any three elements x, y, z of M with $x < y < z$, $x + y$ does not divide z. Determine the largest possible size of M.

Hong Kong 2006

61 For a positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. Define $f_{n+1} = f_1 \circ f_n$. Evaluate $f_{2007}(2^{2006})$.

Hong Kong 2006

Chapter 3

Year 2005

1 Find all positive integers n with the following property: For every positive divisor d of n, $d+1$ divides $n+1$.

Serbia and Montenegro 2005

2 Let A and b be positive integers and $K = \sqrt{\frac{a^2+b^2}{2}}$, $A = \frac{a+b}{2}$. If $\frac{K}{A}$ is a positive integer, prove that $a = b$.

Serbia and Montenegro 2005

3 Determine all polynomials p with real coefficients for which $p(0) = 0$ and

$$f(f(n)) + n = 4f(n) \quad \text{for all } n \in \mathbb{N},$$

where $f(n) = [p(n)]$.

Serbia and Montenegro 2005

4 Let $P(x, y)$ and $Q(x, y)$ be polynomials with integer coefficients. Given integers a_0, b_0, define the sequence of points $X_n(a_n, b_n)_{n\geq 0}$ by $a_{n+1} = P(a_n, b_n)$ and $b_{n+1} = Q(a_n, b_n)$. Suppose that $X_1 \neq X_0$, but $X_k = X_0$ for some $k \in \mathbb{N}$. Show that the number of lattice points on the segment X_nX_{n+1} is the same for each n.

Japan 2005

5 The function $\psi : \mathbb{N} \to \mathbb{N}$ is defined by $\psi(n) = \sum_{k=1}^{n} \gcd(k, n)$.

1. Prove that $\psi(mn) = \psi(m)\psi(n)$ for every two coprime $m, n \in \mathbb{N}$.
2. Prove that for each $a \in \mathbb{N}$ the equation $\psi(x) = ax$ has a solution.

Italy 2005

6 Suppose that $f : \{1, 2, \ldots, 1600\} \to \{1, 2, \ldots, 1600\}$ satisfies $f(1) = 1$ and

$$f^{2005}(x) = x \quad \text{for } x = 1, 2, \ldots, 1600.$$

1. Prove that f has a fixed point different from 1.
2. Find all $n > 1600$ such that any $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ satisfying the above condition has at least two fixed points.

Italy 2005

7 Show that 2005^{2005} is a sum of two perfect squares, but not a sum of two perfect cubes.

Ireland 2005

8 Let x be an integer and y, z, w be odd positive integers. Prove that 17 divides $x^{y^{z^w}} - x^{y^z}$.

Ireland 2005

9 Find the first digit to the left and the first digit to the right of the decimal point in the expansion of $\left(\sqrt{2} + \sqrt{5}\right)^{2000}$.

Ireland 2005

10 Suppose that m and n are odd integers such that $m^2 - n^2 + 1$ divides $n^2 - 1$. Prove that $m^2 - n^2 + 1$ is a perfect square.

Ireland 2005

11 Find all sequences $x_1, \ldots, x_n$ of distinct positive integers such that

$$\frac{1}{2} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2}.$$

HIBMC 2005

12 Does there exist a sequence of 2005 consecutive positive integers that contains exactly 25 prime numbers?

HIBMC 2005

13 Let F_n be the n-th Fibonacci number (where $F_1 = F_2 = 1$). Consider the functions

$$\begin{aligned} f_n(x) &= ||\dots|||x| - F_n| - F_{n-1}| - \dots - F_2| - F_1|, \\ g_n(x) &= |\dots||x-1|-1|-\dots-1| \qquad (F_1 + \dots + F_n \text{ one's}). \end{aligned}$$

Show that $f_n(x) = g_n(x)$ for every real number x.

HIBMC 2005

14 Let (a, b, c) be a Pythagorean triple, i.e. a triplet of positive integers with $a^2 + b^2 = c^2$.

1. Prove that $(\frac{c}{a} + \frac{c}{b})^2 > 8$.
2. Prove that there are no integer n and Pythagorean triple (a, b, c) satisfying $(\frac{c}{a} + \frac{c}{b})^2 = n$.

CANADA 2005

15 Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \le b \le c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

1. Determine all ordered triples (if any) which are n-powerful for all $n \ge 1$.
2. Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

CANADA 2005

16 Find all triples of positive integers (x, y, z) for which

$$\sqrt{\frac{2005}{x+y}} + \sqrt{\frac{2005}{x+z}} + \sqrt{\frac{2005}{y+z}}$$

is an integer.

BULGARIA 2005

17 Let M be the set of rational numbers in the interval $(0, 1)$. Is there a subset A of M such that every element of M can be uniquely represented as a sum of finitely many distinct elements of A?

BULGARIA 2005

18 Suppose that a, b, c are positive integers such that ab divides $c(c^2 - c + 1)$ and $c^2 + 1$ divides $a + b$. Prove that one of the numbers a, b is equal to c and the other one is equal to $c^2 - c + 1$.

BULGARIA 2005

19 A natural number is *palindromic* if writing its (decimal) digits in the reverse order yields the same number. For instance, numbers 481184, 131 and 2 are palindromic. Find all pairs of positive integers (m, n) such that $\underbrace{11\dots1}_{m}\cdot\underbrace{11\dots1}_{n}$ is palindromic.

BRAZIL 2005

20 Given positive integers a, c and an integer b, prove that there exists a positive integer x such that

$$a^x + x \equiv b \pmod{c}.$$

BRAZIL 2005

21 Find all triples of natural numbers (x, y, n) such that $\frac{x!+y!}{n!} = 3^n$.

VIETNAM 2005

22 Lyosha wrote down the numbers $1, 2, \dots, 22^2$ in the cells of a 22×22 table using each number exactly once. Can Oleg always choose a pair of cells sharing a side or a vertex such that the sum of the numbers in these cells is divisible by 4?

RUSSIA 2005

23 Ten distinct nonzero numbers are such that for any two of these numbers, either their sum or their product is rational. Prove that the squares of all these numbers are rational.

RUSSIA 2005

24 Let $S(M)$ denote the sum of the elements of a set M. In how many ways can one partition the numbers $2^0, 2^1, 2^2, \dots, 2^{2005}$ into two nonempty subsets A and B so that the equation $x^2 - S(A)x + S(B) = 0$ has integer roots?

RUSSIA 2005

25 Find the smallest natural number that is not representable in the form $\frac{2^a-2^b}{2^c-2^d}$, where a, b, c, d are natural numbers.

RUSSIA 2005

26 Suppose that positive integers x, y satisfy the equation $2x^2 - 1 = y^{15}$. Prove that if $x > 1$ then x is divisible by 5.

RUSSIA 2005

27 Positive integers x, y, z with $x > 2$ and $y > 1$ satisfy $x^y + 1 = z^2$. Denote by p the number of distinct prime divisors of x and by q that of y. Show that $p \geq q + 2$.

RUSSIA 2005

28 Find all positive integers x, y such that $3^x = 2^x y + 1$

ROMANIA 2005

29 Let n be a positive integer and X be a set of n^2+1 positive integers with the property that every $(n+1)$-element subset of X contains two distinct elements one of which divides the other one. Prove that there are distinct elements $x_1, x_2, \ldots, x_n$ of X such that $x_i \mid x_{i+1}$ for $i = 1, \ldots, n$.

ROMANIA 2005

30 Let m and n be coprime positive integers with m even and n odd. Prove that the sum

$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\left[\frac{mk}{n}\right]} \left\{ \frac{mk}{n} \right\}$$

does not depend on m and n

ROMANIA 2005

31 If $n \geq 0$ is an integer and $p \equiv 7 (mod 8)$ a positive prime number, prove that

$$\sum_{k=1}^{p-1} \left\{ \frac{k^{2^n}}{p} + \frac{1}{2} \right\} = \frac{p-1}{2}.$$

ROMANIA 2005

32 Find all positive integers n for which $n^n + 1$ and $(2n)^{2n} + 1$ are prime numbers.

POLAND 2005

33 The polynomial $W(x) = x^2 + ax + b$ with integer coefficients has the following property: For every prime number p there is an integer k such that both $W(k)$ and $W(k+1)$ are divisible by p. Show that there is an integer m such that $W(m) = W(m+1) = 0$.

POLAND 2005

34 Find all triples (x, y, n) of positive integers satisfying the equation

$$(x - y)^n = xy.$$

POLAND 2005

35 Let $k > 1$ be an integer, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

has all of its terms relatively prime to m

Poland 2005

36 Find all positive integers k such that the product of the decimal digits of k equals $\frac{25}{8}k - 211$.

NrdMO 2005

37 Prove that among any 18 consecutive positive integers not exceeding 2005 there is at least one divisible by the sum of its digits.

Italy 2005

38 Determine all $n \geq 3$ for which there are n positive integers $a_1, \ldots, a_n$ any two of which have a common divisor greater than 1, but any three of which are coprime. Assuming that, moreover, the numbers a_i are less than 5000, find the greatest possible n.

Italy 2005

39 Show that there exist infinitely many square-free positive integers n such that n divides $2005^n - 1$.

Hong Kong 2005

40 Let p be a prime number and n be a positive integer. Show that if q is any positive divisor of $(n+1)^p - n^p$, then $q - 1$ is divisible by p.

Baltic Way 2005

41 The sequence $(x_n)_{n \geq 0}$ is defined by $x_0 = a$, $x_1 = 2$ and

$$x_n = 2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1 \quad \text{for } n > 1.$$

Find all integers a for which $2x_{3n} - 1$ is a perfect square for all $n \geq 1$.

Baltic Way 2005

42 Let x and y be positive integers for which $z = \frac{4xy}{x+y}$ is an odd integer. Prove that z has a positive divisor of the form $4n-1$, $n \in \mathbb{N}$.

Baltic Way 2005

43 Is it possible to find 2005 different positive integers the sum of whose squares is also a square?

Baltic Way 2005

44 Find all positive integers $n = p_1 p_2 \cdots p_k$ which divide $(p_1+1)(p_2+1)\cdots(p_k+1)$, where $p_1 p_2 \cdots p_k$ is the factorization of n into prime factors (not necessarily distinct).

Baltic Way 2005

45 Find all primes p such that $p^2 - p + 1$ is a perfect cube.

BMO 2005

46 Let $n \geq 2$ be an integer, and let S be a subset of $\{1, 2, \ldots, n\}$ such that S neither contains two coprime elements, nor does it contain two elements, one of which divides the other. What is the maximum possible number of elements of S?

BMO 2005

47 Show that there exist infinitely many multiples of 2005 in which each of the decimal digits $0, 1, 2, \ldots, 9$ occurs equally many times.

Austria 2005

Chapter 4

Year 2004

1 Find all pairs of positive integers (a, b) such that $5a^b - b = 2004$.

Serbia and Montenegro 2004

2 Suppose that a, b, c are positive numbers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is an integer. Show that abc is a perfect cube.

Serbia and Montenegro 2004

3 The sequence (a_n) is determined by $a_1 = 0$ and

$$(n+1)^3 a_{n+1} = 2n^2(2n+1)a_n + 2(3n+1) \quad \text{for } n \geq 1.$$

Prove that infinitely many terms of the sequence are positive integers.

Serbia and Montenegro 2004

4 The sequence (a_n) is given by $a_1 = x \in \mathbb{R}$ and $3a_{n+1} = a_n + 1$ for $n \geqslant 1$. Set

$$A = \sum_{n=1}^{\infty} \left[a_n - \frac{1}{6} \right], \quad B = \sum_{n=1}^{\infty} \left[a_n + \frac{1}{6} \right].$$

Compute the sum $A + B$ in terms of x.

Serbia and Montenegro 2004

5 Find all triples (x, y, z) of positive integers such that

$$(x+y)(1+xy) = 2^z.$$

Vietnam 2004

6 Find the least positive integer k with the following property: In each k-element subset of $A = \{1, 2, \ldots, 16\}$ there exist two distinct elements a and b such that $a^2 + b^2$ is a prime number.

VIETNAM 2004

7 Let $S(n)$ be the sum of decimal digits of a natural number n. Find the least value of $S(m)$ if m is an integral multiple of 2003.

VIETNAM 2004

8 Prove that the is no positive integer n for which $2n^2 + 1$, $3n^2 + 1$ and $6n^2 + 1$ are all perfect squares.

JAPAN 2004

9 Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.$$

CANADA 2004

10 Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements of a subset S of T such that no element in S divides any other element in S?

CANADA 2004

11 For every positive integer n, let us write $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ in the form $\frac{p_n}{q_n}$, where p_n and q_n are coprime positive integers.

1. Show that p_{67} is not divisible by 3.
2. Determine all positive integers n for which p_n is divisible by 3.

BULGARIA 2004

12 Assume that a, b, c, d are positive integers such that the number of pairs (x, y) with $0 < x, y < 1$ such that both $ax + by$ and $cx + dy$ are integers equals 2004. Given that $\gcd(a, c) = 6$, determine $\gcd(b, d)$.

BULGARIA 2004

13 Let p be a prime number and let $0 \leq a_1 < \cdots < a_m < p$ and $0 \leq b_1 < \cdots < b_n < p$ be arbitrary integers. Let k denote the number of different residues modulo p the sums $a_i + b_j$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ can give. Prove that

1. if $m + n > p$, then $k = p$;
2. if $m + n \leq p$, then $k \geq m + n - 1$.

Bulgaria 2004

14 Consider the sequence (a_n) given by $a_0 = a_1 = a_2 = a_3 = 1$ and

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2.$$

Prove that all its terms are integers.

Brazil 2004

15 Show that there exist infinitely many pairs of positive integers (m, n) such that $\binom{m}{n-1} = \binom{m-1}{n}$.

Brazil 2004

16 Let $(x+1)^p (x-3)^q = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$, where p, q are positive integers.

1. Prove that if $a_1 = a_2$, then $3n$ is a perfect square.
2. Prove that there exist infinitely many pairs (p, q) for which $a_1 = a_2$.

Brazil 2004

17 The sequence (L_n) is givn by $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$. Prove that if a prime number p divides $L_{2k} - 2$ for $k \in \mathbb{N}$, then p also divides $L_{2k+1} - 1$.

Brazil 2004

18 Find all solutions in the set of prime numbers of the equation

$$x^y - y^x = xy^2 - 19.$$

BMO 2004

19 Find all positive integers that can be written in the form $\frac{a^2+ab+b^2}{ab-1}$ for some positive integers a, b not both equal to 1.

Romania 2004

20 Given integers a, b, c with b odd, define a sequence (x_n) by $x_0 = 4$, $x_1 = 0$, $x_2 = 2c$, $x_3 = 3b$ and

$$x_{n+3} = ax_{n-1} + bx_n + cx_{n+1}.$$

Prove that $p \mid x_{p^m}$ for all primes p and positive integers m.

Romania 2004

21 Prove that for all positive integers m, n with m odd it holds that

$$3^m n \mid \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

Romania 2004

22 Let $m, n \in \mathbb{N}$, $m > 1$. Suppose that $a^m - 1$ is divisible by n for all a coprime to n. Prove that

$$n \leq 4m(2^m - 1).$$

Romania 2004

23 Let p be a prime number and $f(x) = \sum_{i=1}^{p-1} a_i x^{i-1}$ be a polynomial with $a_i = 1$ if i is a square modulo p and $a_i = -1$ otherwise.

1. Prove that $f(x) \equiv x - 1$ modulo $(x-1)^2$ if $p \equiv 3 \pmod 4$;
2. Prove that $f(x) \equiv 0$ modulo $(x-1)^2$ if $p \equiv 5 \pmod 8$.

Romania 2004

24 Find all integers $n > 1$ for which $2^2 + 3^2 + \cdots + n^2$ is a power of a prime.

Poland 2004

25 Determine whether there exists an infinite sequence $a_1, a_2, \ldots$ of positive integers satisfying $\frac{1}{a_n} = \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}}$ for all $n \in \mathbb{N}$.

Poland 2004

26 Consider the functions $f(x) = 2^x$ and $g(x) = f(f(f(f(f(f(f(x)))))))$ (the seventh iteration of f). Show that the number $g(3) - g(0)$ is divisible by $g(2) - g(0)$.

Poland 2004

27 Find all positive integers n which have exactly $\sqrt{n}$ positive divisors.

Poland 2004

28 An integer $m > 1$ is given. The infinite sequence $x_0, x_1, x_2, \ldots$ is defined by

$$x_i = \begin{cases} 2^i & \text{for } i < m, \\ x_{i-1} + x_{i-2} + \cdots + x_{i-m} & \text{for } i \geq m. \end{cases}$$

Find the largest natural number k for which there exist k successive terms of this sequence which are divisible by m.

Poland 2004

29 The Fibonacci sequence is defined by $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Prove that there is a strictly increasing arithmetic progression whose no term is in the Fibonacci sequence.

NrdMO 2004

30 Given a finite sequence $x_{1,1}, x_{2,1}, \ldots, x_{n,1}$ of integers ($n \geq 2$), not all equal, define the sequences $x_{1,k}, \ldots, x_{n,k}$ by

$$x_{i,k+1} = \frac{1}{2}(x_{i,k} + x_{i+1,k}), \quad \text{where } x_{n+1,k} = x_{1,k}.$$

Show that if n is odd, then not all $x_{j,k}$ are integers. Is this also true for even n?

NrdMO 2004

31 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,

$$(2^m + 1)f(n)f(2^m n) = 2^m f(n)^2 + f(2^m n)^2 + (2^m - 1)^2 n.$$

Italy 2004

32 A positive integer n is said to be a *perfect power* if $n = a^b$ for some integers a, b with $b > 1$.

1. Find 2004 perfect powers which form an arithmetic progression.
2. Prove that perfect powers cannot form an infinite arithmetic progression.

Italy 2004

33

1. Is 2005^{2004} the sum of two perfect squares?
2. Is 2004^{2005} the sum of two perfect squares?

Italy 2004

34 Let $S = \{1, 2, \ldots, 100\}$. Find the number of functions $f : S \to S$ satisfying the following conditions:

(i) $f(1) = 1$;

(ii) f is bijective;

(iii) $f(n) = f(g(n))f(h(n))$ for all $n \in S$, where $g(n)$ and $h(n)$ are the positive integers with $g(n) \le h(n)$ and $g(n)h(n) = n$ that minimize $h(n) - g(n)$. (For instance, $g(80) = 8$, $h(80) = 10$.)

Hong Kong 2004

35 The function f is defined for each integer k by

$$f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k,$$

where $(k)_{2n+1}$ denotes the multiple of $2n+1$ closest to k. Find the set of values taken by f.

Baltic Way 2004

36 A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three faces meeting at that vertex. If the sum of these products is 1001, what is the sum of the six numbers on the faces?

Baltic Way 2004

37 Find all sets X consisting of at least two positive integers such that for every $m, n \in X$ with $n > m$ there exists $k \in X$ with $n = mk^2$.

Baltic Way 2004

38 Prove that for every nonconstant polynomial $f(x)$ with integer coefficients there exists an integer n such that $f(n)$ has at least 2004 distinct prime factors.

Baltic Way 2004

39 A set S of $n-1$ natural numbers is given ($n \geq 3$), not all of which are congruent modulo n. Prove that it is possible to choose a non-empty subset of S with the sum of elements divisible by n.

Baltic Way 2004

40 Is there an infinite sequence of prime numbers $p_1, p_2, \ldots$ satisfying $|p_{n+1} - 2p_n| = 1$ for each $n \in \mathbb{N}$?

Baltic Way 2004

41 For natural numbers a, b, define $Z(a,b) = \frac{(3a)!(4b)!}{a!^4 b!^3}$.

1. Prove that $Z(a,b)$ is an integer for $a \leq b$.
2. Prove that for each natural number b there are infinitely many natural numbers a such that $Z(a,b)$ is not an integer.

Austria 2004

42 Each of the $2N = 2004$ real numbers $x_1, x_2, \ldots, x_{2004}$ equals either $\sqrt{2}-1$ or $\sqrt{2}+1$. Can the sum $\sum_{k=1}^{N} x_{2k-1}x_{2k}$ take the value 2004? Which integral values can this sum take?

Austria 2004

43

1. Given any set $\{p_1, p_2, \ldots, p_k\}$ of prime numbers, show that the sum of the reciprocals of all numbers of the form $p_1^{r_1} \cdots p_k^{r_k}$ ($r_1, \ldots, r_k \in \mathbb{N}$) is also a reciprocal of an integer.
2. Compute the above sum, knowing that $1/2004$ occurs among the summands.
3. Prove that for each k-element set $\{p_1, \ldots, p_k\}$ of primes ($k > 2$), the above sum is smaller than $1/N$, where $N = 2 \cdot 3^{k-2}(k-2)!$.

Austria 2004

44 Show that there is an infinite sequence $a_1, a_2, \ldots$ of natural numbers such that $a_1^2 + a_2^2 + \cdots + a_N^2$ is a perfect square for all N. Give a recurrent formula for one such sequence.

Austria 2004

Chapter 5

Year 2003

1 Find the number of solutions to the equation

$$x_1{}^4 + x_2{}^4 + \ldots + x_{10}{}^4 = 2011$$

in the set of positive integers.

Serbia and Montenegro 2003

2 A subset S of $\mathbb{N}$ has the following properties:

(i) Among any 2003 consecutive natural numbers, at least one is in S;

(ii) If $n \in S$ and $n > 1$, then $[n/2] \in S$ as well.

Prove that $S = \mathbb{N}$.

Serbia and Montenegro 2003

3 Prove that the number $\left[(5+\sqrt{35})^{2n-1}\right]$ is divisible by 10^n for each $n \in \mathbb{N}$.

Serbia and Montenegro 2003

4 Let n be an even number and S be the set of all arrays of 0 and 1 of length n. Prove that S can be partitioned into disjoint three-element subsets such that: for any three arrays $(a_i), (b_i), (c_i)$ in the same subset and all $i = 1, 2, \ldots, n$, the number $a_i + b_i + c_i$ is even.

Serbia and Montenegro 2003

5 Find the greatest positive integer n for which the system

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2$$

has an integer solution $(x, y_1, \ldots, y_n)$.

VIETAM 2003

6 Find all three-digit numbers n which are equal to the number formed by three last digits of n^2.

ITALY 2003

7 Let n be a positive integer. Show that there exist three distinct integers between n^2 and $n^2 + n + 3\sqrt{n}$, such that one of them divides the product of the other two.

HIBMC 2003

8 Find the last three digits of the number $2003^{2002^{2001}}$.

CANADA 2003

9 Determine the smallest prime number which divides $x^2 + 5x + 23$ for some integer x.

BRAZIL 2003

10 Does there exist a set B of 4004 distinct natural numbers, such that for any subset A of B containing 2003 elements, the sum of the elements of A is not divisible by 2003?

BMO 2003

11 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ which satisfy the following conditions:

(i) $f(x+y) - yf(x) - xf(y) = f(x)f(y) - x - y + xy$ for all $x, y \in \mathbb{Q}$;

(ii) $f(x) = 2f(x+1) + 2 + x$ for all $x \in \mathbb{Q}$;

(iii) $f(1) + 1 > 0$.

BMO 2003

12 Suppose that M is a set of 2003 numbers such that, for any distinct $a, b \in M$, the number $a^2 + b\sqrt{2}$ is rational. Prove that $a\sqrt{2}$ is rational for all $a \in M$.

RUSSIA 2003

13 Suppose that M is a set of 2003 numbers such that, for any distinct $a, b, c \in M$, the number $a^2 + bc$ is rational. Prove that there is a natural number n such that $a\sqrt{n}$ is rational for all $a \in M$.

Russia 2003

14 Let a_0 be a natural number. The sequence (a_n) is defined by $a_{n+1} = a_n/5$ is a_n is divisible by 5, and $a_{n+1} = \left[\sqrt{5}a_n\right]$ otherwise. Show that the sequence a_n is increasing starting from some term.

Russia 2003

15 Is it possible to write a natural number in every cell of an infinite chessboard in such a manner that for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$?

Russia 2003

16 Let f be an irreducible monic polynomial with integer coefficients, such that $|f(0)|$ is not a perfect square. Prove that the polynomial $g(x) = f(x^2)$ is also irreducible over non-constant polynomials with integer coefficients.

Romania 2003

17 Find all integers a, b, m, n, where $m > n > 0$, such that the polynomial $f(x) = x^n + ax + b$ divides the polynomial $g(x) = x^m + ax + b$

Romania 2003

18 Let $d(n)$ denote the sum of decimal digits of a positive integer n. Prove that for each $k \in \mathbb{N}$ there exists a positive integer m such that the equation $x + d(x) = m$ has exactly k solutions in $\mathbb{N}$.

Romania 2003

19 Decide whether there exist a prime p and nonnegative integers x, y, z such that $(12x+5)(12y+7) = p^z$.

Poland 2003

20 Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational x, y

$$f(x^2 + y) = xf(x) + f(y).$$

Poland 2003

21 Find all positive integer solutions of the equation $a^2 + b^2 = c^2$ such that a and c are prime and b is a product of at most four prime numbers.

POLAND 2003

22 Let be given nonconstant polynomials $W_1(x), W_2(x), \ldots, W_n(x)$ with integer coefficients. Prove that for some integer a all the numbers $W_1(a), W_2(a), \ldots, W_n(a)$ are composite.

POLAND 2003

23 Find all polynomials W with real coefficients having the following property: If $x + y$ is a rational number, then so is $W(x) + W(y)$.

POLAND 2003

24 Show that for each prime $p > 3$ there exist integers x, y, k with $0 < 2k < p$ such that $kp + 3 = x^2 + y^2$.

POLAND 2003

25 Find all polynomials W with integer coefficients satisfying the following condition: For every natural number n, $2^n - 1$ is divisible by $W(n)$.

POLAND 2003

26 A prime number p and integers x, y, z with $0 < x < y < z < p$ are given. Show that if the numbers x^3, y^3, z^3 give the same remainder when divided by p, then $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

POLAND 2003

27 Find all triples (x, y, z) of integers satisfying the equation

$$x^3 + y^3 + z^3 - 3xyz = 2003.$$

NRDMO 2003

28 Find all triples (a, b, p) with a, b positive integers and p a prime number such that $2^a + p^b = 19^a$.

ITALY 2003

29 Let $p(x)$ be a polynomial with integer coefficients and let n be an integer. Suppose that there is a positive integer k for which $f^{(k)}(n) = n$, where $f^{(k)}(x)$ is the polynomial obtained as the composition of k polynomials f. Prove that $p(p(n)) = n$.

Italy 2003

30 Determine all integers a, b, c such that

$$\frac{1}{2}(a+b)(b+c)(c+a) + (a+b+c)^3 = 1 - abc.$$

Hong Kong 2003

31 Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which for all $x \in \mathbb{Q}^+$ fulfill

$$f\left(\frac{1}{x}\right) = f(x) \quad \text{and} \quad \left(1 + \frac{1}{x}\right) f(x) = f(x+1).$$

Baltic Way 2003

32 Find all pairs of positive integers (a, b) such that $a - b$ is a prime number and ab is a perfect square.

Baltic Way 2003

33 All the positive divisors of a positive integer n are stored into an increasing array. Mary is writing a program which decides for an arbitrarily chosen divisor $d > 1$ whether it is a prime. Let n have k divisors not greater than d. Mary claims that it suffices to check divisibility of d by the first $\lceil k/2 \rceil$ divisors of n: d is prime if and only if none of them but 1 divides d. Is Mary right?

Baltic Way 2003

34 Every integer is to be colored blue, green, red, or yellow. Can this be done in such a way that if a, b, c, d are not all 0 and have the same color, then $3a - 2b \neq 2c - 3d$?

Baltic Way 2003

35 Let a and b be positive integers. Show that if $a^3 + b^3$ is the square of an integer, then $a + b$ is not a product of two different prime numbers.

Baltic Way 2003

36 Suppose that the sum of all positive divisors of a natural number n, n excluded, plus the number of these divisors is equal to n. Prove that $n = 2m^2$ for some integer m.

Baltic Way 2003

37 Find all triples of prime numbers (p, q, r) such that $p^q + p^r$ is a perfect square.

Austria 2003

38 Consider the polynomial $P(n) = n^3 - n^2 - 5n + 2$. Determine all integers n for which $P(n)^2$ is a square of a prime.

Austria 2003

39 Prove that, for any integer $g > 2$, there is a unique three-digit number $\overline{abc}_g$ in base g whose representation in some base $h = g \pm 1$ is $\overline{cba}_h$.

Austria 2003

Chapter 6

Year 2002

1 Find all pairs (n, k) of positive integers such that $\binom{n}{k} = 2002$.

Serbia and Montenegro 2002

2 Let m and n be positive integers. Prove that the number $2^n - 1$ is divisible by $(2^m - 1)^2$ if and only if n is divisible by $m(2^m - 1)$.

Serbia and Montenegro 2002

3 Is there a positive integer k such that none of the digits $3, 4, 5, 6$ occurs in the decimal representation of the number $2002! \cdot k$?

Serbia and Montenegro 2002

4 Determine all positive integers n for which the equation

$$x + y + u + v = n\sqrt{xyuv}$$

has a solution in positive integers x, y, u, v.

Vietnam 2002

5 Find all three-digit numbers which are equal to 34 times the sum of their digits.

Italy 2002

6 Determine the values of n for which all the solutions of the equation $x^3 - 3x + n = 0$ are integers.

Italy 2002

7 Prove that if n is a positive integer such that $m = 5^n + 3^n + 1$ is prime, then n is divisible by 12.

ITALY 2002

8 Find all triples of positive integers (p, q, n), with p and q primes, satisfying

$$p(p+3) + q(q+3) = n(n+3).$$

IRELAND 2002

9 The sequence (a_n) is defined by $a_1 = a_2 = a_3 = 1$ and

$$a_{n+1}a_{n-2} - a_n a_{n-1} = 2 \quad \text{for all } n \geq 3.$$

Prove that a_n is a positive integer for all $n \geq 1$.

IRELAND 2002

10 Suppose n is a product of four distinct primes a, b, c, d such that

(i) $a + c = d$;

(ii) $a(a + b + c + d) = c(d - b)$;

(iii) $1 + bc + d = bd$.

Determine n.

IRELAND 2002

11 Let $\alpha = 2 + \sqrt{3}$. Prove that $\alpha^n - [\alpha^n] = 1 - \alpha^{-n}$ for all $n \in \mathbb{N}_0$.

IRELAND 2002

12 Find the greatest exponent k for which 2001^k divides $2000^{2001^{2002}} + 2002^{2001^{2000}}$.

HIBMC 2002

13 Let $p \geq 5$ be a prime number. Prove that there exists a positive integer $a < p - 1$ such that neither of $a^{p-1} - 1$ and $(a+1)^{p-1} - 1$ is divisible by p^2.

HIBMC 2002

14 Let $p(x)$ be a polynomial with rational coefficients, of degree at least 2. Suppose that a sequence (r_n) of rational numbers satisfies $r_n = p(r_{n+1})$ for every $n \geq 1$. Prove that the sequence (r_n) is periodic.

HIBMC 2002

15 We call a positive integer n *practical* if every positive integer less than or equal to n can be written as the sum of distinct divisors of n. Prove that the product of two practical numbers is also practical.

Canada 2002

16 Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$xf(y) + yf(x) = (x+y)f(x^2+y^2) \qquad \text{for all } x, y \in \mathbb{N}_0.$$

Canada 2002

17 Show that there exists a set A of positive integers with the following properties:

1. A has 2002 elements;
2. The sum of any number of distinct elements of A (at least one) is never a perfect power (i.e. a number of the form a^b, where $a, b \in \mathbb{N}$ and $b \geq 2$).

Brazil 2002

18 Find all triples (a, b, c) of nonnegative integers such that $2^c - 1$ divides $2^a + 2^b + 1$.

APMC 2002

19 Consider the set $A = \{2, 7, 11, 13\}$. A polynomial f with integer coefficients has the property that for each integer n, $f(n)$ is divisible by some prime from A. Prove that there exists $p \in A$ such that $p \mid f(n)$ for all integers n.

APMC 2002

20 Find all functions $f : \mathbb{N} \to \mathbb{R}$ satisfying $f(x + 22) = f(x)$ and $f(x^2 y) = f(x)^2 f(y)$ for all positive integers x and y.

APMC 2002

21 From the interval $(2^{2n}, 2^{3n})$ are selected $2^{2n-1} + 1$ odd numbers. Prove that there are two among the selected numbers, none of which divides the square of the other.

Russia 2002

22 Prove that for every integer $n > 10000$ there exists an integer m such that it can be written as the sum of two squares, and $0 < m - n < 3\sqrt[4]{n}$.

Russia 2002

23 Determine the smallest natural number which can be represented both as the sum of 2002 positive integers with the same sum of decimal digits, and as the sum of 2003 integers with the same sum of decimal digits.

Russia 2002

24 Prove that there exist infinitely many natural numbers n such that the numerator of $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ in the lowest terms is not a power of a prime number.

Russia 2002

25 The sequence (a_n) is defined by

$$a_0 = a_1 = 1 \quad \text{and} \quad a_{n+1} = 14a_n - a_{n-1} \text{ for all } n \geq 1.$$

Prove that $2a_n - 1$ is a perfect square for any $n \geq 0$.

Romania 2002

26 Assume that P and Q are polynomials with coefficients in the set $\{1, 2002\}$ such that P divides Q, prove that then $\deg P + 1$ divides $\deg Q + 1$.

Romania 2002

27 Given $p_0, p_1 \in \mathbb{N}$, define p_{n+2} $(n \geq 0)$ inductively to be the smallest prime divisor of $p_n + p_{n+1}$. Prove that the real number whose decimal representation is given by $x = \overline{0.p_0 p_1 p_2 \ldots}$ is rational.

Romania 2002

28 Let m and n be positive integers, not of the same parity, such that $m < n < 5m$. Show that the set $\{1, 2, \ldots, 4mn\}$ can be partitioned into pairs of numbers so that the sum in each pair is a square.

Romania 2002

29 Find all integers $p \leq q \leq r$ for which all the numbers

$$pq + r,\ pq + r^2,\ qr + p,\ qr + p^2,\ rp + q,\ rp + q^2$$

are prime.

Poland 2002

30 Determine all positive integers a, b, c such that the numbers $a^2 + 1$ and $b^2 + 1$ are prime and $(a^2 + 1)(b^2 + 1) = c^2 + 1$.

Poland 2002

31 Given a positive integer k, the sequence (a_n) is defined by $a_1 = k + 1$ and

$$a_{n+1} = a_n^2 - ka_n + k \quad \text{for } n \geq 1.$$

Show that for $m \neq n$ the numbers a_m, a_n are relatively prime.

Poland 2002

32 Find all pairs of positive integers x, y such that $(x + y)^2 - 2(xy)^2 = 1$.

Poland 2002

33 A positive integer n_1 contains 333 decimal digits, and all these digits are nonzero. For $i = 1, 2, \ldots, 332$, set n_{i+1} to be the number obtained from n_i by moving the last digit of n_i to the beginning. Prove that 333 divides either none, or all of the numbers $n_1, n_2, \ldots, n_{333}$.

Poland 2002

34 Let p be a prime number such that $p \equiv 1 \pmod 4$. Determine $\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^2}{p} \right\}$, where $\{x\} = x - [x]$.

Hong Kong 2002

35 Find all nonnegative integers m for which $a_m = \left(2^{2m+1}\right)^2 + 1$ is divisible by at most two different primes.

Baltic Way 2002

36 Show that the sequence $\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \ldots$ is periodic modulo 2002.

Baltic Way 2002

37 Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3 - 1)(n^2 - 1)$.

Baltic Way 2002

38 Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

has no solutions in positive rational numbers.

Baltic Way 2002

39 Does there exist an infinite non-constant arithmetic progression, each term of which is of the form a^b, where a and b are positive integers with $b \geq 2$

Baltic Way 2002

40 The sequence (a_n) is defined by $a_1 = 20$, $a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for every $n \geq 1$. Find all positive integers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

BMO 2002

41 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers n

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$

BMO 2002

42 Determine all integers a and b such that

$$(19a + b)^{18} + (a + b)^{18} + (a + 19b)^{18}$$

is a perfect square.

Austria 2002

Chapter 7

Year 2001

1 Let be given a positive integer n and two coprime integers a, b greater than 1. Let p and q be two odd divisors of $a^{6^n} + b^{6^n}$ different from 1. Find the remainder of $p^{6^n} + q^{6^n}$ when divided by $6 \cdot 12^n$.

Vietnam 2001

2 Let $n \geq 1$ be a given integer. Consider a permutation $(a_1, a_2, \dots, a_{2n})$ of the first $2n$ positive integers such that the numbers $|a_{i+1} - a_i|$ are distinct for $i = 1, 2, \dots, 2n-1$. Prove that $a_1 - a_{2n} = n$ if and only if $1 \leq a_{2k} \leq n$ for every $k = 1, 2, \dots, n$.

Vietnam 2001

3 Consider the equation $x^{2001} = y^x$.

1. Find all solutions (x, y) with x prime and y a positive integer.
2. Find all solutions (x, y) in positive integers.

(Recall that $2001 = 3 \cdot 23 \cdot 29$.)

Italy 2001

4 A positive integer is called *monotone* if has at least two digits and all its digits are nonzero and appear in a strictly increasing or strictly decreasing order.

1. Compute the sum of all monotone five-digit numbers.
2. Find the number of final zeros in the least common multiple of all monotone numbers (with any number of digits).

Italy 2001

5 Find all positive integer solutions (a, b, c, n) of the equation

$$2^n = a! + b! + c!.$$

IRELAND 2001

6 Show that if an odd prime number p can be expressed in the form $x^5 - y^5$ for some integers x, y, then

$$\sqrt{\frac{4p+1}{5}} = \frac{v^2+1}{2} \quad \text{for some odd integer } v.$$

IRELAND 2001

7 Find the least positive integer a such that 2001 divides $55^n + a \cdot 32^n$ for some odd n.

IRELAND 2001

8 Find all nonnegative real numbers x for which

$$\sqrt[3]{13+\sqrt{x}} + \sqrt[3]{13-\sqrt{x}}$$

is an integer.

IRELAND 2001

9 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy

$$f(x + f(y)) = f(x) + y \quad \text{for all } x, y \in \mathbb{N}.$$

IRELAND 2001

10 Find positive integers x, y, z such that $x > z > 1999 \cdot 2000 \cdot 2001 > y$ and $2000x^2 + y^2 = 2001z^2$.

HIBMC 2001

11 Let be given 32 positive integers with the sum 120, none of which is greater than 60. Prove that these integers can be divided into two disjoint subsets with the same sum of elements.

HIBMC 2001

12 Let be given an integer $a_0 > 1$. We define a sequence $(a_n)_{n\geq 1}$ in the following way. For every $k \geq 0$, a_{k+1} is the least integer $x > a_k$ such that $(x, a_0 a_1 \cdots a_k) = 1$. Determine for which values of a_0 are all the members a_k of the sequence primes or powers of primes.

BRAZIL 2001

13 Let $f(n)$ be the least positive integer k such that n divides $1+2+\cdots+k$. Prove that $f(n) = 2n-1$ if and only if n is a power of 2.

BRAZIL 2001

14 Let k be a fixed positive integer. Consider the sequence defined by $a_0 = 1$ and

$$a_{n+1} = a_n + \left[\sqrt[k]{a_n}\right], \quad n = 0, 1, \ldots$$

For each k find the set A_k of all integer values of the sequence $\sqrt[k]{a_n}$, $n \geq 0$.

APMC 2001

15 Consider the set A of all positive integer containing no zero (decimal) digit and which are divisible by their sum of digits.

1. Prove that A contains infinitely many numbers whose decimal expansion contains each of its digits the same number of times.

2. Show that for each $k \in \mathbb{N}$ there is a k-digit number in A.

APMC 2001

16 The integers from 1 to 999999 are partitioned into two groups: the first group consists of those integers for which the closest perfect square is odd, whereas the second group consists of those for which the closest perfect square is even. In which group is the sum of the elements greater?

RUSSIA 2001

17 Find all odd integers $n > 1$ such that, whenever a and b are coprime divisors of n, the number $a+b-1$ is also a divisor of n.

RUSSIA 2001

18 Let a and b be two distinct natural numbers such that $ab(a+b)$ is divisible by $a^2 + ab + b^2$. Prove that $|a-b| > \sqrt[3]{ab}$

RUSSIA 2001

19 Let $k, n > 1$ be integers such that the number $p = 2k - 1$ is prime. Prove that, if the number $\binom{n}{2} - \binom{k}{2}$ is divisible by p, then it is divisible by p^2.

POLAND 2001

20 Find all integers $n \geq 3$ for which the following statement is true:
Any arithmetic progression $a_1, \ldots, a_n$ with n terms for which $a_1 + 2a_2 + \cdots + na_n$ is rational contains at least one rational term.

POLAND 2001

21 Suppose that a and b are integers such that $2^n a + b$ is a perfect square for all $n \in \mathbb{N}$. Show that $a = 0$.

POLAND 2001

22 Show that the number

$$\sum_{n=0}^{10^{10}} \binom{2 \cdot 10^{10}}{2n} 5^n$$

is divisible by $2^{2 \cdot 10^{10} - 1}$.

POLAND 2001

23 Prove that for each positive integer k there exists a positive integer m such that each of the numbers $m, 2m, 3m, \ldots, m^2$ has exactly k one's in the binary expansion.

POLAND 2001

24 Let $S(n)$ denote the sum of digits of a natural number n. Prove that for each n the number $S(2n^2 + 3)$ is not a perfect square.

POLAND 2001

25 Let x, y, z be positive integers with $xy = z^2 + 1$. Prove that there exist integers a, b, c, d such that

$$x = a^2 + b^2, \quad y = c^2 + d^2, \quad z = ac + bd.$$

IRAN 2001

26 Find, with proof, all positive integers n such that the equation

$$x^3 + y^3 + z^3 = nx^2y^2z^2$$

has a solution in positive integers.

HONG KONG 2001

27 Let $k \geq 4$ be an integer. Prove that if $P(x)$ is a polynomial with integer coefficients such that $0 \leq F(c) \leq k$ for $c = 0, 1, \ldots, k+1$, then $F(0) = F(1) = \cdots = F(k+1)$.

Hong Kong 2001

28 A function $f : \mathbb{N} \to \mathbb{R}$ is such that for all $n > 1$ there exists a prime divisor p of n such that $f(n) = f(\frac{n}{p}) - f(p)$. Given that $f(2001) = 1$, what is the value of $f(2002)$?

Baltic Way 2001

29 Let n be a positive integer. Prove that one can choose no less than $2^{n-1} + n$ numbers from the set $\{1, 2, \ldots, 2^n\}$ such that for any two different chosen numbers x, y, $x + y$ does not divide xy.

Baltic Way 2001

30 Let a be an odd integer. Prove that $a^{2^n} + 2^{2^n}$ and $a^{2^m} + 2^{2^m}$ are coprime for all positive integers $n \neq m$.

Baltic Way 2001

31 What is the smallest positive odd integer having the same number of positive divisors as 360?

Baltic Way 2001

32 From a quadruple of integers (a, b, c, d) each of the sequences

$$(c, d, a, b), \quad (b, a, d, c), \quad (a + nc, b + nd, c, d), \quad (a + nb, b, c + nd, d)$$

for an arbitrary integer n can be obtained by one step. Is it possible to obtain $(3, 4, 5, 7)$ from $(1, 2, 3, 4)$ through a sequence of such steps?

Baltic Way 2001

33 Let n be a positive integer. Prove that if a, b are integers greater than 1 such that $ab = 2^n - 1$, then the number $ab - (a - b) - 1$ is of the form $k \cdot 2^{2m}$, where k is odd and m a positive integer.

BMO 2001

34 Determine all integers m for which all solutions of the equation $3x^3 - 3x^2 + m = 0$ are rational.

Austria 2001

35 Find all pairs of integers (m, n) such that

$$\left|(m^2 + 2000m + 999999) - (3n^3 + 9n^2 + 27n)\right| = 1.$$

Austria 2001

Chapter 8

Year 2000

1 Consider the polynomial $P(x) = x^3 + 153x^2 - 111x + 38$.

1. Prove that there are at least nine integers a in the interval $[1, 3^{2000}]$ for which $P(a)$ is divisible by 3^{2000}.
2. Find the number of integers a in $[1, 3^{2000}]$ with the property from (a).

Vietnam 2000

2 Prove or disprove: For any positive integer k there exists an integer $n > 1$ such that the binomial coefficient $\binom{n}{i}$ is divisible by k for any $1 \leq i \leq n - 1$.

HIBMC 2000

3 For a given integer d, let us define $S = \{m^2 + dn^2 \mid m, n \in \mathbb{Z}\}$. Suppose that p, q are two elements of S, where p is prime and $p \mid q$. Prove that $r = \frac{q}{p}$ also belongs to S.

HIBMC 2000

4 The sequence (a_n) is defined by $a_1 = 43$, $a_2 = 142$ and $a_{n+1} = 3a_n + a_{n-1}$ for $n \geq 2$. Prove that

1. a_n and a_{n+1} are coprime for all n;
2. for every $m \in \mathbb{N}$ there are infinitely many natural numbers n such that $a_n - 1$ and $a_{n+1} - 1$ are both divisible by m.

Bulgaria 2000

5 Let p be a prime number and let $a_1, a_2, \ldots, a_{p-2}$ be positive integers such that p does not divide a_k or $a_k^k - 1$ for any k. Prove that the product of some of the a_i's is congruent to 2 modulo p.

Bulgaria 2000

6 Let p be a prime number and let $a_1, a_2, \ldots, a_{p-2}$ be positive integers such that p does not divide a_k or $a_k^k - 1$ for any k. Prove that the product of some of the a_i's is congruent to 2 modulo p.

BULGARIA 2000

7 For a positive integer n, let A_n be the set of all positive numbers greater than 1 and less than n which are coprime to n. Find all n such that all the elements of A_n are prime numbers.

BRAZIL 2000

8 For a positive integer n, let $V(n, b)$ be the number of decompositions of n into a product of one or more positive integers greater than b. For example, $36 = 6 \cdot 6 = 4 \cdot 9 = 3 \cdot 12 = 3 \cdot 3 \cdot 4$, so that $V(36, 2) = 5$. Prove that for all positive integers n, b it holds that

$$V(n, b) < \frac{n}{b}.$$

BRAZIL 2000

9 Let $\sigma(n)$ denote the sum of all positive divisors of a positive integer n (for example, $\sigma(6) = 1 + 2 + 3 + 6 = 12$). We call a number n *quasi-perfect* if $\sigma(n) = 2n - 1$. Let $n \bmod k$ denote the remainder of n upon division by k, and $s(n) = \sum_{k=1}^{n} (n \bmod k)$ (for example, $s(6) = 0 + 0 + 0 + 2 + 1 + 0 = 3$). Prove that n is quazi-perfect if and only if $s(n) = s(n - 1)$.

BRAZIL 2000

10 Define a function f on the set of positive integers in the following way. If n is written as $2^a(2b+1)$ for integers a and b, then $f(n) = a^2 + a + 1$. Find the minimum positive n for which

$$f(1) + f(2) + \cdots + f(n) \geq 123456.$$

BRAZIL 2000

11 A positive integer is a *power* if it is of the form t^s for some integers $t, s \geq 2$. Prove that for any natural number n there exists a set A of positive integers with the following properties:

(i) A has n elements;

(ii) Every element of A is a power;

(iii) For any $2 \leq k \leq n$ and any $r_1, \ldots, r_k \in A$, $\frac{r_1 + \cdots + r_k}{k}$ is a power.

BMO 2000

12 Find all polynomials $P(x)$ with real coefficients having the following property: There exists a positive integer n such that the equality

$$\sum_{k=1}^{2n+1} (-1)^k \left[\frac{k}{2}\right] P(x+k) = 0$$

holds for infinitely many real numbers x.

APMC 2000

13 Find all positive integers N possessing only 2 and 5 as prime divisors, such that $N+25$ is a square.

APMC 2000

14 The sequence $a_1 = 1, a_2, a_3, \ldots$ is defined as follows: if $a_n - 2$ is a natural number not already occurring on the board, then $a_{n+1} = a_n - 2$; otherwise $a_{n+1} = a_n + 3$. Prove that every nonzero perfect square occurs in the sequence as the previous term increased by 3.

Russia 2000

15 Evaluate the sum $\left[\frac{2^0}{3}\right] + \left[\frac{2^1}{3}\right] + \left[\frac{2^2}{3}\right] + \cdots + \left[\frac{2^{1000}}{3}\right]$.

Russia 2000

16 A perfect number, greater than 6, is divisible by 3. Prove that it is also divisible by 9. (A natural number is *perfect* if the sum of its proper divisors is equal to the number itself: e.g. $6 = 1 + 2 + 3$.)

Russia 2000

17 Prove that one can partition the set of natural numbers into 100 nonempty subsets such that among any three natural numbers a, b, c satisfying $a + 99b = c$, there are two that belong to the same subset.

Russia 2000

18 Let be given a sequence of nonnegative integers $a_1, a_2, \ldots, a_n$. For $k = 1, 2, \ldots, n$, denote

$$m_k = \max_{1 \le l \le k} \frac{a_{k-l+1} + a_{k-l+2} + \cdots + a_k}{l}.$$

Prove that for every $\alpha > 0$ the number of values of k for which $m_k > \alpha$ is less than $fraca_1 + a_2 + \cdots + a_n\alpha$.

Russia 2000

19 Solve in integers the equation $x^{2000} + 2000^{1999} = x^{1999} + 2000^{2000}$.

POLAND 2000

20 Prove that for all integers $n \geq 2$ and all prime numbers p the number $n^{p^p} + p^p$ is composite.

POLAND 2000

21 Prove that among any 12 consecutive integers there is one that cannot be written as a sum of 10 fourth powers.

POLAND 2000

22 An n-tuple $(c_1, c_2, \dots, c_n)$ of positive integers is called *admissible* if each positive integer k not exceeding $2(c_1 + c_2 + \cdots + c_n)\}$ can be represented in the form

$$k = \sum_{i=1}^{n} a_i c_i, \quad \text{with } a_i \in \{-2, -1, 0, 1, 2\}.$$

For each n find the maximum possible value of $c_1 + \cdots + c_n$ if $(c_1, \dots, c_n)$ is admissible.

POLAND 2000

23 Does there exist a natural number N which is a power of 2, such that one can permute its decimal digits to obtain a different power of 2?

IRAN 2000

24 A sequence of natural numbers $c_1, c_2, \dots$ is called *perfect* if every natural number m with $1 \leq m \leq c_1 + \cdots + c_n$ can be represented as

$$m = \frac{c_1}{a_1} + \frac{c_2}{a_2} + \cdots + \frac{c_n}{a_n}, \quad a_i \in \mathbb{N}.$$

Given n, find the maximum possible value of c_n in a perfect sequence (c_i).

IRAN 2000

25 Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function that satisfies $f(1) = 1$ and

$$f(n+1) = \begin{cases} f(n) + 2 & \text{if } n = f(f(n) - n + 1), \\ f(n) + 1 & \text{otherwise.} \end{cases}$$

1. Prove that $f(f(n) - n + 1)$ is either n or $n + 1$.
2. Determine f.

IRAN 2000

26 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

(i) $f(m) = 1$ if and only if $m = 1$;

(ii) If $d = (m, n)$, then $f(mn) = \frac{f(m)f(n)}{f(d)}$;

(iii) $f^{2000}(m) = m$ for every $m \in \mathbb{N}$.

Iran 2000

27 Prove that for every natural number n there exists a polynomial $p(x)$ with integer coefficients such that $p(1), p(2), \ldots, p(n)$ are distinct powers of 2.

Iran 2000

28 Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

Ireland 2000

29 For each positive integer n find all positive integers m for which there exist positive integers $x_1 < x_2 < \cdots < x_n$ with

$$\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} = m.$$

Ireland 2000

30 Show that in each set of ten consecutive integers there is one that is coprime with each of the other integers. (For example, in the set $\{114, 115, \ldots, 123\}$ there are two such numbers: 119 and 121.)

Ireland 2000

31 Find all prime numbers p and q such that $\frac{(7^p-2^p)(7^q-2^q)}{pq}$ is an integer.

Hong Kong 2000

32 Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression of integers such that $i \mid a_i$ for $i = 1, 2, \ldots, n-1$ and $n \nmid a_n$. Prove that n is a prime power.

Baltic Way 2000

33 Find all positive integers n having exactly $n/100$ positive divisors.

Baltic Way 2000

34 Let n be a positive integer not divisible by 2 or 3. Prove that for all integers k, the number $(k+1)^n - k^n - 1$ is divisible by $k^2 + k + 1$.

Baltic Way 2000