1 PEN I10

Show that for all primes p,

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p+1)(p-1)(p-2)}{4}.$$

First Solution. For $1 \le k \le p-1$, we have $k^3 \not\equiv 0 \mod p$ and $(p-k)^3 \equiv -(k^3) \mod p$, and therefore

$$\left(\frac{k^3}{p} - \left\lfloor \frac{k^3}{p} \right\rfloor\right) + \left(\frac{(p-k)^3}{p} - \left\lfloor \frac{(p-k)^3}{p} \right\rfloor\right) = 1.$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor &= \sum_{k=1}^{p-1} \frac{k^3}{p} - \sum_{k=1}^{p-1} \left(\frac{k^3}{p} - \left\lfloor \frac{k^3}{p} \right\rfloor\right) \\
&= \frac{1}{p}\left(\sum_{k=1}^{p-1} k^3\right) - \frac{1}{2}\sum_{k=1}^{p-1}\left[\left(\frac{k^3}{p} - \left\lfloor \frac{k^3}{p} \right\rfloor\right) + \left(\frac{(p-k)^3}{p} - \left\lfloor \frac{(p-k)^3}{p} \right\rfloor\right)\right] \\
&= \frac{1}{p}\left(\frac{p(p-1)}{2}\right)^2 - \frac{p-1}{2} \\
&= \frac{(p-2)(p-1)(p+1)}{4}.
\end{aligned}$$

□

Second Solution. Motivated by this first proof, we show a natural generalization:

Proposition 1. *Let p be an odd prime and let q be an integer that is not divisible by p. If $f : \mathbb{Z}_+^* \to \mathbb{R}$ is a function such that:*

i) $\frac{f(k)}{p}$ is not an integer, for any $k = 1, 2, \ldots, p-1$;

ii) $f(k) + f(p-k)$ is an integer divisible by p, for any $k = 1, 2, \ldots, p-1$,

then

$$\sum_{k=1}^{p-1} \left\lfloor \frac{qf(k)}{p} \right\rfloor = \frac{q}{p} \cdot \sum_{k=1}^{p-1} f(k) - \frac{p-1}{2}.$$

Proof. From *ii*) it follows that

$$\frac{qf(k)}{p} + \frac{qf(p-k)}{p} \in \mathbb{Z}.$$

From *i*) we obtain that $\frac{qf(k)}{p} \notin \mathbb{Z}$ and $\frac{qf(p-k)}{p} \notin \mathbb{Z}$, for any $k = 1, 2, \ldots, p-1$, and hence

$$0 < \left\{\frac{qf(k)}{p}\right\} + \left\{\frac{qf(p-k)}{p}\right\} < 2.$$

On other hand,

$$\left\{\frac{qf(k)}{p}\right\} + \left\{\frac{qf(p-k)}{p}\right\} = \left(\frac{qf(k)}{p} + \frac{qf(p-k)}{p}\right) - \left(\left\lfloor \frac{qf(k)}{p} \right\rfloor + \left\lfloor \frac{qf(k)}{p} \right\rfloor\right),$$

and thus, because $\frac{qf(k)}{p} + \frac{qf(p-k)}{p} \in \mathbb{Z}$, we deduce that

$$\left\{\frac{qf(k)}{p}\right\} + \left\{\frac{qf(p-k)}{p}\right\} \in \mathbb{Z}.$$

It now follows that

$$\left\{\frac{qf(k)}{p}\right\}+\left\{\frac{qf(p-k)}{p}\right\}=1,\quad \text{for any } k=1,\ 2,\ \ldots,\ p-1.$$

Summing up and dividing by 2 yields

$$\sum_{k=1}^{p-1}\left\{\frac{qf(k)}{p}\right\}=\frac{p-1}{2},$$

and therefore,

$$\sum_{k=1}^{p-1}\frac{qf(k)}{p}-\sum_{k=1}^{p-1}\left\lfloor\frac{qf(k)}{p}\right\rfloor=\frac{p-1}{2}.$$

This proves Proposition 1.

Obviously, the function $f(x)=x^3$ satisfies both conditions from Proposition 1. Hence,

$$\begin{aligned}\sum_{k=1}^{p-1}\left\lfloor\frac{k^3}{p}\right\rfloor &= \frac{q}{p}\cdot\sum_{k=1}^{p-1}k^3-\frac{p-1}{2}\\ &= \frac{q}{p}\cdot\left(\frac{(p-1)p}{2}\right)^2-\frac{p-1}{2}\\ &= \frac{(p-2)(p-1)(p+1)}{4}.\end{aligned}$$

□

We now proceed with another two applications of Proposition 1. We begin with Gauss' celebrated formula.

Proposition 2. *(Gauss) Let p and q be two relatively prime integers. The following identity holds:*

$$\sum_{k=1}^{p-1}\left\lfloor\frac{qk}{p}\right\rfloor=\frac{(p-1)(q-1)}{2}.$$

Proof. The function $f(x)=x$ satisfies both conditions in Proposition 1. Hence,

$$\begin{aligned}\sum_{k=1}^{p-1}\left\lfloor\frac{qk}{p}\right\rfloor &= \frac{q}{p}\cdot\sum_{k=1}^{p-1}k-\frac{p-1}{2}\\ &= \frac{q}{p}\cdot\frac{(p-1)p}{2}-\frac{p-1}{2}\\ &= \frac{(p-1)(q-1)}{2}.\end{aligned}$$

□

Proposition 3. *Let p be an odd prime. Show that*

$$\sum_{k=1}^{p-1}\frac{k^p-k}{p}\equiv\frac{p+1}{2}\mod p.$$

Proof. The function $f(x) = \frac{x^p}{p}$ satisfies both conditions in Proposition 1. Thus, by setting $q = 1$ (note that we are allowed to do that), we get

$$\begin{aligned}
\sum_{k=1}^{p-1} \left\lfloor \frac{k^p}{p^2} \right\rfloor &= \frac{1}{p} \cdot \sum_{k=1}^{p-1} \frac{k^p}{p} - \frac{p-1}{2} \\
&= \frac{1}{p} \cdot \sum_{k=1}^{p-1} \frac{k^p}{p} - \frac{1}{p^2} \cdot \sum_{k=1}^{p-1} k + \frac{1}{p^2} \cdot \frac{(p-1)p}{2} - \frac{p-1}{2} \\
&= \frac{1}{p} \cdot \sum_{k=1}^{p-1} \frac{k^p - k}{p} - \frac{1}{p} \cdot \frac{(p-1)^2}{2}.
\end{aligned}$$

Thus,

$$\sum_{k=1}^{p-1} \frac{k^p - k}{p} - \frac{(p-1)^2}{2} = p \cdot \sum_{k=1}^{p-1} \left\lfloor \frac{k^p}{p^2} \right\rfloor,$$

and therefore

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{k^p - k}{p} &\equiv \frac{(p-1)^2}{2} \mod p \\
&\equiv \frac{p^2+1}{2} \mod p \\
&\equiv \frac{p+1}{2} \mod p.
\end{aligned}$$

□