0.1 Exponential Congruence Sequence

1 D5 Prove that for $n \geq 2$,

PEN D5
D6

$$\underbrace{2^{2^{\cdots^{2}}}}_{n\text{ terms}} \equiv \underbrace{2^{2^{\cdots^{2}}}}_{n-1\text{ terms}} \pmod{n}. \tag{1}$$

D6 Show that, for any fixed integer $n \geq 1$ the sequence

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \cdots \pmod{n} \tag{2}$$

is eventually constant.

First Solution. We can prove stronger statement by induction over n. First, write $a_n = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n\text{times}}$

Proposition 0.1.1. $a_m \pmod{n}$ *is constant for all* $m \geq n-1$.

Proof. Base case $n = 1$ is obvious.
If n is even, write $n = 2^s t$. Then the sequence is clearly constant after $n-1$ modulo 2^s. It is also true for modulo t by the induction hypothesis.
So we can suppose that n is odd. Then there is a constant c satisfying

$$a_m \equiv c \pmod{\phi(n)} \tag{3}$$

for all m after $\phi(n) - 1 (\geq n-2)$. Thus $a_m \equiv 2^c \pmod{n} (m \leq n-1)$ by Euler Theorem, as desired. □

This can prove a following corollary.

Corollary 0.1.1. $\underbrace{2^{2^{\cdots^{2}}}}_{n\text{ terms}} - \underbrace{2^{2^{\cdots^{2}}}}_{n-1\text{ terms}}$ *is divisible by all positive integer less or equal than* n.

□

The form $2^{a_n} = a_{n+1}$ looks like ISL2006 N9. This can also lead to a little bit different problem.

Proposition 0.1.2. *Prove that for arbitrary positive integer* n*, There exists an integer* m *such that*

$$\frac{2^m - m}{n} \tag{4}$$

is integer.

Proof. Using the previous proposition, there is a constant c satisfying

$$\underbrace{2^{2^{\cdots^{2}}}}_{n\text{ terms}} \equiv c \pmod{n\phi(n)}. \tag{5}$$

By the definition, $2^c \equiv \underbrace{2^{2^{\cdots^{2}}}}_{n+1\text{ terms}} \equiv c \pmod{n}$. So c can be a desired m. □

We can replace "2" in this problem into any positive integer. And this proof works on other equations such as $a_n = 3 \cdot 2^{a_{n-1}}$ or $a_n = 2^{a_{n-1}} + 3^{a_{n-1}}$ or $a_n = 2^{2^{a_{n-1}}}$ (replace $\phi(n)$ into $\phi(\phi(n))$). From this, we get a generalized corollary.

Corollary 0.1.2. *For an arbitrary positive integer n, there exists an integer m such that $f(m)$ is integer, where $f(n)$ is*

$$\frac{a \cdot b^m - m}{n} \tag{6}$$

or

$$\frac{a_1^m + \cdots + a_k^m - m}{n} \tag{7}$$

or

$$\frac{2^{2^m} - m}{n} \tag{8}$$

for integers $a, b, a_1, \cdots, a_k$.

Proof is same as Corollary 2.5.1. However this method doesn't work for Original Shortlisted problem.
First 50 constants for $a_n = 2^{a_{n-1}}$ are 0, 0, 1, 0, 1, 4, 2, 0, 7, 6, 9, 4, 3, 2, 1, 0, 1, 16, 5, 16, 16, 20, 6, 16, 11, 16, 7, 16, 25, 16, 2, 0, 31, 18, 16, 16, 9, 24, 16, 16, 18, 16, 4, 20, 16, 6, 17, 16, 23, 36.
16 appears very often in first 200 constants and I think this would last more.