PROBLEMS IN ELEMENTARY NUMBER THEORY 1 (2008) NO. 1

Problems in English

1. *Let* $\mathbb{N} = \{1, 2, 3, \cdots\}$ *denote the set of positive integers. Find all functions* $f : \mathbb{N} \to \mathbb{N}$ *such that for all* $m, n \in \mathbb{N}$*:* $f(2) = 2$, $f(mn) = f(m)f(n)$, $f(n+1) > f(n)$.

[K12] Canada 1969

2. *Prove that there are no integers* x *and* y *satisfying* $x^2 = y^5 - 4$.

[H15] Balkan Mathematical Olympiad 1998

3. *Suppose the set* $M = \{1, 2, \ldots, n\}$ *is partitioned into* t *disjoint subsets* $M_1, \ldots, M_t$. *Show that if* $n \geq \lfloor t! \cdot e \rfloor$ *then at least one class* M_z *contains three elements* x_i, x_j, x_k *with the property that* $x_i - x_j = x_k$.

[O53] Schur Theorem

4. *Let* p *be a prime number of the form* $4k + 1$. *Show that*

$$\sum_{i=1}^{p-1} \left(\left\lfloor \frac{2i^2}{p} \right\rfloor - 2 \left\lfloor \frac{i^2}{p} \right\rfloor \right) = \frac{p-1}{2}.$$

[I11] Korea 2000

5.
(a) *Let* $d(n)$ *denote the number of positive divisors of the number* n. *Prove that the sequence* $d(n^2 + 1)$ *does not become strictly monotonic from some point onwards.*
(b) *Prove that* $d((n^2 + 1)^2)$ *does not become monotonic from any given point onwards.*

[J11] Saint-Peterburg, 1998

6. *Let* a *and* b *be positive integers such that* $ab + 1$ *divides* $a^2 + b^2$. *Show that*

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

[A3] IMO 1988/6

7. *Suppose that p is an odd prime. Prove that*

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

[D2] Putnam 1991/B4

8. *Let n be a prime and $a_1 < a_2 < \ldots < a_n$ be integers. Prove that $a_1, a_2, \ldots, a_n$ is an arithmetic progression if and only if there exists a partition of $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$ into n sets $A_1, A_2, \ldots, A_n$ so that*

$$a_1 + A_1 = a_2 + A_2 = \ldots = a_n + A_n,$$

where $x + A = \{x + a | a \in A\}$.

[O35] Romania TST 1998

9. *Suppose that m does not have a primitive root. Show that*

$$a^{\frac{\varphi(m)}{2}} \equiv 1 \pmod{m}$$

for every a relatively prime to m.

[B6]

10. *Consider the set of all five-digit numbers whose decimal representation is a permutation of the digits $1, 2, 3, 4, 5$. Prove that this set can be divided into two groups, in such a way that the sum of the squares of the numbers in each group is the same.*

[O49] D. Fomin, [Ams, pp. 12]