0.1 A Hidden Divisibility

1 (AMM, Problem E2510, Saul Singer) Show that for all prime numbers p,

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$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

First Solution. It is natural to search for a closed form for $Q(p)$. We see immediately that

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1} = \left(\frac{\prod_{k=1}^{p-1} k^k}{[(p-1)!]^{(p+1)/2}} \right)^2 .$$

Now, the procedure is standard. Using, for example, Legendre's formula (though often atributed to De Polignac; see http://mathworld.wolfram.com/LegendresFormula.html) we can now evaluate the exponent of an arbitrary prime number p in $((p-1)!)^{p+1}$ and prove that it is smaller than the exponent of the same prime p in $\prod_{k=1}^{p-1} k^{2k}$. For a more detailed explanation, we invite the reader to read the second post from [1].

□

Though the problem appeared in the MONTHLY, as indicated in the cited source, the nice result presented here was given also as a contest problem in the Romanian IMO Team Selection Tests from 1988. It was then suggested by Laurenţiu Panaitopol, who had up is sleeve a magical solution.

Second Solution. We look again (more carefully though) at

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1} = \left(\frac{\prod_{k=1}^{p-1} k^k}{[(p-1)!]^{(p+1)/2}} \right)^2 .$$

We see lots of factorials, but apparently there's nothing we can do with them. So, let's try to avoid them. Here comes the idea. Let's look at

$$Q'(p) = \prod_{k=1}^{p-1} \frac{\binom{p}{k}}{p}.$$

What do we have here? Note that by developing the binomials we get

$$Q'(p) = \prod_{k=1}^{p-1} \frac{\binom{p}{k}}{p} = \frac{[(p-1)!]^{p-1}}{\left(\prod_{k=1}^{p-1} k!\right)\left(\prod_{k=1}^{p-1} (p-k)!\right)} = \frac{[(p-1)!]^{p-1}}{\left(\prod_{k=1}^{p-1} k!\right)^2}.$$

Now we can see the light. Doesn't this last expression look like $Q(p)$? Yes, it does. Just note that the exponents of each $k \in \overline{1, p-1}$ in $[(p-1)!]^{p-1}$, $\left(\prod_{k=1}^{p-1} k!\right)^2$ are $p-1$ and $2(p-k)$, respectively. This means that

$$\begin{aligned} Q'(p) &= \frac{[(p-1)!]^{p-1}}{\left(\prod_{k=1}^{p-1} k!\right)^2} = \prod_{k=1}^{p-1} k^{2k-p-1} \\ &= Q(p). \end{aligned}$$

Thus, we have settled that

$$Q(p) = \prod_{k=1}^{p-1} \frac{\binom{p}{k}}{p},$$

which is obviously an integer, according to the fact that p divides $\binom{p}{k}$, for all $k \in \overline{1, p-1}$. $\square$

References

1 *PEN Problem A13*, http://www.mathlinks.ro/viewtopic.php?t=150381