Number Theory Problems from IMO Shortlist 1999 – 2006

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There’ll be plenty of time to rest in the grave. - Paul Erdős.
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1 Determine all pairs \((x, y)\) of integers such that
\[
1 + 2^x + 2^{2x+1} = y^2.
\]
For $x \in (0,1)$ let $y \in (0,1)$ be the number whose $n$-th digit after the decimal point is the $2^n$-th digit after the decimal point of $x$. Show that if $x$ is rational then so is $y$. 

Solution
We define a sequence \((a_1, a_2, a_3, \ldots)\) by setting

\[a_n = \frac{1}{n} \left( \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor \right)\]

for every positive integer \(n\).

a) Prove that there is an infinite number of positive integers \(n\) such that \(a_{n+1} > a_n\).

b) Prove that there is an infinite number of positive integers \(n\) such that \(a_{n+1} < a_n\).
Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let $k$ be a positive integer. Consider the polynomial $Q(x) = P(P(\cdots P(P(x))\cdots))$, where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ such that $Q(t) = t$. 

\textbf{Solution}
Prove that the equation \( x^7 - 1 = y^5 - 1 \) does not have integer solutions.
Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer $c$, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers $x$ and $y$ such that $ax + by = c$. An integer $c$ is called a local champion if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$. Find all local champions and determine their number.

**Solution**
For all positive integers $n$, show that there exists a positive integer $m$ such that $n$ divides $2^m + m$.

Remark. (Brazil '05) Given positive integers $a, c$ and integer $b$, prove that there exists a positive integer $x$ such that $a^x + x \equiv b \pmod{c}$.
Chapter 2

46th IMO Mexico 2005

Determine all positive integers relatively prime to all the terms of the infinite sequence

\[ a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1. \]

Solution
Let $a_1, a_2, \cdots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer $n$ the numbers $a_1, a_2, \cdots, a_n$ leave $n$ different remainders upon division by $n$. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \cdots$. 

SOLUTION
Let \( a, b, c, d, e, f \) be positive integers and let \( S = a + b + c + d + e + f \). Suppose that the number \( S \) divides \( abc + def \) and \( ab + bc + ca - de - ef - df \). Prove that \( S \) is composite.
Find all $n$ such that there exists a unique integer $a$ such that $0 \leq a < n!$ with the following property:

$$n! \mid a^n + 1.$$
Denote by $d(n)$ the number of divisors of the positive integer $n$. A positive integer $n$ is called highly divisible if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers $m$ and $n$ with $m < n$ are called consecutive if there exists no highly divisible integer $s$ satisfying $m < s < n$.

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form $(a, b)$ with $a | b$.

(b) Show that for every prime number $p$ there exist infinitely many positive highly divisible integers $r$ such that $pr$ is also highly divisible.
Let $a$, $b$ be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers $n$. Prove that $a = b$. 

Solution
Let \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \), where \( a_0, \cdots, a_n \) are integers, \( a_n > 0 \), \( n \geq 2 \).

Prove that there exists a positive integer \( m \) such that \( P(m!) \) is a composite number.
Let $\tau(n)$ denote the number of positive divisors of the positive integer $n$. Prove that there exist infinitely many positive integers $a$ such that the equation $\tau(an) = n$ does not have a positive integer solution $n$. 

**Solution**
The function $f$ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$f(n) = \sum_{k=1}^{n} \gcd(k, n), \quad n \in \mathbb{N}.$$ 

a) Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.

b) Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.

c) Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

**Solution**
Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying \[ f(m)^2 + f(n) | (m^2 + n)^2 \]
for all positive integers $m$ and $n$. 

**SOLUTION**
Let \( k \) be a fixed integer greater than 1, and let \( m = 4k^2 - 5 \). Show that there exist positive integers \( a \) and \( b \) such that the sequence \((x_n)\) defined by

\[ x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n, \quad n = 0, 1, 2, \ldots \]

has all of its terms relatively prime to \( m \).
We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers $n$ such that $n$ has a multiple which is alternating.

**Solution**
Given an integer \( n > 1 \), denote by \( P_n \) the product of all positive integers \( x \) less than \( n \) and such that \( n \) divides \( x^2 - 1 \). For each \( n > 1 \), find the remainder of \( P_n \) on division by \( n \).
Let $p$ be an odd prime and $n$ a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length $p^n$. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by $p^{n+1}$.
Let $x_0, x_1, x_2, \cdots$ be the sequence defined by

\[
x_i = 2^i, \quad 0 \leq i \leq 2003,
\]

\[
x_i = \sum_{j=1}^{2004} x_{i-j}, \quad i \geq 2004.
\]

Find the greatest $k$ for which the sequence contains $k$ consecutive terms divisible by 2004.

**Solution**
Each positive integer \( a \) is subjected to the following procedure, yielding the number \( d = d(a) \): (a) The last digit of \( a \) is moved to the first position. The resulting number is called \( b \). (b) The number \( b \) is squared. The resulting number is called \( c \). (c) The first digit of \( c \) is moved to the last position. The resulting number is called \( d \). (All numbers are considered in the decimal system.) For instance, \( a = 2003 \) gives \( b = 3200 \), \( c = 10240000 \) and \( d = 02400001 = 2400001 = d(2003) \). Find all integers \( a \) such that \( d(a) = a^2 \).

**Solution**
24 Determine all pairs of positive integers \((a, b)\) such that
\[
\frac{a^2}{2ab^2 - b^3 + 1}
\]
is a positive integer.

SOLUTION
Let $b$ be an integer greater than 5. For each positive integer $n$, consider the number

$$x_n = \overline{11 \cdots 122 \cdots 25, \overline{n-1}}_n,$$

written in base $b$. Prove that the following condition holds if and only if $b = 10$: there exists a positive integer $M$ such that for any integer $n$ greater than $M$, the number $x_n$ is a perfect square.

**Solution**
An integer $n$ is said to be good if $|n|$ is not the square of an integer. Determine all integers $m$ with the following property: $m$ can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.
Let $p$ be a prime number. Prove that there exists a prime number $q$ such that for every integer $n$, the number $n^p - p$ is not divisible by $q$. 

SOLUTION
The sequence \(a_0, a_1, a_2, \cdots\) is defined as follows:

\[ a_0 = 2, \quad a_{k+1} = 2a_k^2 - 1, \quad k \geq 0. \]

Prove that if an odd prime \(p\) divides \(a_n\), then \(2^{n+3}\) divides \(p^2 - 1\).
Let $p$ be a prime number and let $A$ be a set of positive integers that satisfies the following conditions: (1) the set of prime divisors of the elements in $A$ consists of $p - 1$ elements; (2) for any nonempty subset of $A$, the product of its elements is not a perfect $p$-th power. What is the largest possible number of elements in $A$?

**Solution**
Chapter 5

43rd IMO United Kingdom 2002

30  What is the smallest positive integer $t$ such that there exist integers $x_1, x_2, \cdots, x_t$ with

\[ x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002} \? \]

**Solution**
Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than $n^2$, and determine when it is a divisor of $n^2$.

**Solution**
Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least $4^n$ divisors.

**Solution**
33 Is there a positive integer $m$ such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a + b + c}$$

has infinitely many solutions in positive integers $a, b, c$?

**SOLUTION**
Let \( m, n \geq 2 \) be positive integers, and let \( a_1, a_2, \ldots, a_n \) be integers, none of which is a multiple of \( m^{n-1} \). Show that there exist integers \( e_1, e_2, \ldots, e_n \), not all zero, with \( |e_i| < m \) for all \( i \), such that \( e_1 a_1 + e_2 a_2 + \cdots + e_n a_n \) is a multiple of \( m^n \).
Find all pairs of positive integers \( m, n \geq 3 \) for which there exist infinitely many positive integers \( a \) such that

\[
\frac{a^m + a - 1}{a^n + a^2 - 1}
\]

is itself an integer.
Chapter 6

42nd IMO United States of America 2001

36 01N01 Prove that there is no positive integer $n$ such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n + k)!$ equals $k$.

SOLUTION
Consider the system \( x + y = z + u, \ 2xy = zu \). Find the greatest value of the real constant \( m \) such that \( m \leq \frac{x}{y} \) for any positive integer solution \((x, y, z, u)\) of the system, with \( x \geq y \).
Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and $a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|$, $n \geq 4$. Determine $a_{14^{14}}$. 

**Solution**
39 Let $p \geq 5$ be a prime number. Prove that there exists an integer $a$ with $1 \leq a \leq p - 2$ such that neither $a^{p-1} - 1$ nor $(a + 1)^{p-1} - 1$ is divisible by $p^2$. 

SOLUTION
Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

**SOLUTION**
Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?

**Solution**
Chapter 7

41st IMO Korea 2000

42. Determine all positive integers $n \geq 2$ that satisfy the following condition: For all integers $a$ and $b$ relatively prime to $n$,

$$a \equiv b \pmod{n}$$

if and only if

$$ab \equiv 1 \pmod{n}.$$
For every positive integer $n$ let $d(n)$ the number of all positive integers of $n$. Determine all positive integers $n$ with the property: $d^3(n) = 4n$. 

**Solution**
Does there exist a positive integer $n$ such that $n$ has exactly 2000 prime divisors and $n$ divides $2^n + 1$?
Find all triplets of positive integers \((a, m, n)\) such that \(a^m + 1 \mid (a + 1)^n\).
46  Prove that there exist infinitely many positive integers $n$ such that $p = nr$, where $p$ and $r$ are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

Solution
Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.
Find all the pairs of positive integers \((x, p)\) such that \(p\) is a prime, \(x \leq 2p\) and \(x^{p-1}\) is a divisor of \((p - 1)^x + 1\).
49. Prove that every positive rational number can be represented in the form \( \frac{a^3+b^3}{c^3+d^3} \) where \( a, b, c, d \) are positive integers.

Solution
Prove that there exists two strictly increasing sequences \((a_n)\) and \((b_n)\) such that 
\[ a_n(a_n + 1) \text{ divides } b_n^2 + 1 \text{ for every natural } n. \]

**Solution**
Denote by $S$ the set of all primes such the decimal representation of $\frac{1}{p}$ has the fundamental period divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0, a_1a_2 \cdots a_{3r}, a_1a_2 \cdots a_{3r}, \cdots,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k, p)$ by

$$f(k, p) = a_k + a_{k+r(p)} + a_{k+2r(p)}$$

a) Prove that $S$ is infinite. b) Find the highest value of $f(k, p)$ for $k \geq 1$ and $p \in S$. 

**Solution**
Let $n, k$ be positive integers such that $n$ is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer $m$ which is divisible by $n$ and the sum of its digits in decimal representation is $k$. 

**Solution**
Prove that for every real number $M$ there exists an infinite arithmetic progression such that:

1. each term is a positive integer and the common difference is not divisible by 10;
2. the sum of the digits of each term (in decimal representation) exceeds $M$. 

**Solution**
Chapter 9

All Problems

Problem 1 (06N01 - IMO 06/4). Determine all pairs \((x, y)\) of integers such that

\[1 + 2^x + 2^{2x+1} = y^2.\]

Problem 2 (06N02). For \(x \in (0, 1)\) let \(y \in (0, 1)\) be the number whose \(n\)-th digit after the decimal point is the \(2^n\)-th digit after the decimal point of \(x\). Show that if \(x\) is rational then so is \(y\).

Problem 3 (06N03 - Italy '07). We define a sequence \((a_1, a_2, a_3, \ldots)\) by setting

\[a_n = \frac{1}{n} \left( \left[ \frac{n}{1} \right] + \left[ \frac{n}{2} \right] + \cdots + \left[ \frac{n}{n} \right] \right)\]

for every positive integer \(n\). Hereby, for every real \(x\), we denote by \([x]\) the integral part of \(x\) (this is the greatest integer which is \(\leq x\)).

a) Prove that there is an infinite number of positive integers \(n\) such that \(a_{n+1} > a_n\).

b) Prove that there is an infinite number of positive integers \(n\) such that \(a_{n+1} < a_n\).

Problem 4 (06N04). Let \(P(x)\) be a polynomial of degree \(n > 1\) with integer coefficients and let \(k\) be a positive integer. Consider the polynomial \(Q(x) = P(P(\cdots P(P(x)) \cdots))\), where \(P\) occurs \(k\) times. Prove that there are at most \(n\) integers \(t\) such that \(Q(t) = t\).

Problem 5 (06N05 - Brazil '07). Prove that the equation \(x^7 - 1 = y^5 - 1\) does not have integer solutions.

Problem 6 (06N06). Let \(a > b > 1\) be relatively prime positive integers. Define the weight of an integer \(c\), denoted by \(w(c)\) to be the minimal possible value of \(|x| + |y|\) taken over all pairs of integers \(x\) and \(y\) such that \(ax + by = c\). An integer \(c\) is called a local champion if \(w(c) \geq w(c \pm a)\) and \(w(c) \geq w(c \pm b)\). Find all local champions and determine their number.
Problem 7 (06N07). For all positive integers \( n \), show that there exists a positive integer \( m \) such that \( n \) divides \( 2^m + m \).

Remark. (Brazil '05) Given positive integers \( a, c \) and integer \( b \), prove that there exists a positive integer \( x \) such that \( a^x + x \equiv b \pmod{c} \).

Problem 8 (05N01 - IMO 05/4). Determine all positive integers relatively prime to all the terms of the infinite sequence
\[
a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.
\]

Problem 9 (05N02 - IMO 05/2). Let \( a_1, a_2, \ldots \) be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer \( n \) the numbers \( a_1, a_2, \ldots, a_n \) leave \( n \) different remainders upon division by \( n \). Prove that every integer occurs exactly once in the sequence \( a_1, a_2, \ldots \).

Problem 10 (05N03 - India '06). Let \( a, b, c, d, e, f \) be positive integers and let \( S = a + b + c + d + e + f \). Suppose that the number \( S \) divides \( abc + dcf \) and \( ab + bc + ca - de - ef - df \). Prove that \( S \) is composite.

Problem 11 (05N04 - Iran '06). Find all \( n \) such that there exists a unique integer \( a \) such that \( 0 \leq a < n! \) with the following property:
\[
n! \mid a^n + 1.
\]

Problem 12 (05N05). Denote by \( d(n) \) the number of divisors of the positive integer \( n \). A positive integer \( n \) is called highly divisible if \( d(n) > d(m) \) for all positive integers \( m < n \). Two highly divisible integers \( m \) and \( n \) with \( m < n \) are called consecutive if there exists no highly divisible integer \( s \) satisfying \( m < s < n \).

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form \((a, b)\) with \( a \mid b \).

(b) Show that for every prime number \( p \) there exist infinitely many positive highly divisible integers \( r \) such that \( pr \) is also highly divisible.

Problem 13 (05N06 - Taiwan '06). Let \( a, b \) be positive integers such that \( b^n + n \) is a multiple of \( a^n + n \) for all positive integers \( n \). Prove that \( a = b \).

Problem 14 (05N07). Let \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \), where \( a_0, \ldots, a_n \) are integers, \( a_n > 0 \), \( n \geq 2 \). Prove that there exists a positive integer \( m \) such that \( P(m!) \) is a composite number.
Problem 15 (04N01). Let \( \tau(n) \) denote the number of positive divisors of the positive integer \( n \). Prove that there exist infinitely many positive integers \( a \) such that the equation \( \tau(an) = n \) does not have a positive integer solution \( n \).

Problem 16 (04N02). The function \( f \) from the set \( \mathbb{N} \) of positive integers into itself is defined by the equality

\[
f(n) = \sum_{k=1}^{n} \gcd(k, n), \quad n \in \mathbb{N}.
\]

a) Prove that \( f(mn) = f(m)f(n) \) for every two relatively prime \( m, n \in \mathbb{N} \).

b) Prove that for each \( a \in \mathbb{N} \) the equation \( f(x) = ax \) has a solution.

c) Find all \( a \in \mathbb{N} \) such that the equation \( f(x) = ax \) has a unique solution.

Problem 17 (04N03). Find all functions \( f : \mathbb{N} \rightarrow \mathbb{N} \) satisfying

\[
f(m)^2 + f(n) \mid (m^2 + n)^2
\]

for all positive integers \( m \) and \( n \).

Problem 18 (04N04 - Poland '05). Let \( k \) be a fixed integer greater than 1, and let \( m = 4k^2 - 5 \). Show that there exist positive integers \( a \) and \( b \) such that the sequence \( (x_n) \) defined by

\[x_0 = a, \ x_1 = b, \ x_{n+2} = x_{n+1} + x_n, n = 0, 1, 2, \cdots\]

has all of its terms relatively prime to \( m \).

Problem 19 (04N05 - IMO 04/6 ). We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers \( n \) such that \( n \) has a multiple which is alternating.

Problem 20 (04N06 - Taiwan '05 ). Given an integer \( n > 1 \), denote by \( P_n \) the product of all positive integers \( x \) less than \( n \) and such that \( n \) divides \( x^2 - 1 \). For each \( n > 1 \), find the remainder of \( P_n \) on division by \( n \).

Problem 21 (04N07). Let \( p \) be an odd prime and \( n \) a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length \( p^n \). Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by \( p^{n+1} \).
Problem 22 (03N01 - Singapore ’04). Let \(x_0, x_1, x_2, \cdots\) be the sequence defined by
\[
x_i = 2^i, \quad 0 \leq i \leq 2003,
\]
\[
x_i = \sum_{j=1}^{2004} x_{i-j}, \quad i \geq 2004.
\]
Find the greatest \(k\) for which the sequence contains \(k\) consecutive terms divisible by 2004.

Problem 23 (03N02 - Germany ’04). Each positive integer \(a\) is subjected to the following procedure, yielding the number \(d = d(a)\): (a) The last digit of \(a\) is moved to the first position. The resulting number is called \(b\). (b) The number \(b\) is squared. The resulting number is called \(c\). (c) The first digit of \(c\) is moved to the last position. The resulting number is called \(d\). (All numbers are considered in the decimal system.) For instance, \(a = 2003\) gives \(b = 3200\), \(c = 10240000\) and \(d = 02400001 = 2400001 = d(2003)\). Find all integers \(a\) such that \(d(a) = a^2\).

Problem 24 (03N03 - IMO 03/2). Determine all pairs of positive integers \((a,b)\) such that
\[
\frac{a^2}{2ab^2 - b^4 + 1}
\]
is a positive integer.

Problem 25 (03N04 - Germany ’04). Let \(b\) be an integer greater than 5. For each positive integer \(n\), consider the number
\[
x_n = \frac{11 \cdots 1}{n-1} \frac{22 \cdots 25}{n},
\]
written in base \(b\). Prove that the following condition holds if and only if \(b = 10\): there exists a positive integer \(M\) such that for any integer \(n\) greater than \(M\), the number \(x_n\) is a perfect square.

Problem 26 (03N05 - Moldova ’04). An integer \(n\) is said to be [i]good[/i] if \(|n|\) is not the square of an integer. Determine all integers \(m\) with the following property: \(m\) can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

Problem 27 (03N06 - IMO 06/6). Let \(p\) be a prime number. Prove that there exists a prime number \(q\) such that for every integer \(n\), the number \(n^p - p\) is not divisible by \(q\).

Problem 28 (03N07). The sequence \(a_0, a_1, a_2, \cdots\) is defined as follows:
\[
a_0 = 2, \quad a_{k+1} = 2a_k^2 - 1, \quad k \geq 0.
\]
Prove that if an odd prime \(p\) divides \(a_n\), then \(2^{n+3}\) divides \(p^2 - 1\).
Problem 29 (03N08). Let $p$ be a prime number and let $A$ be a set of positive integers that satisfies the following conditions: (1) the set of prime divisors of the elements in $A$ consists of $p-1$ elements; (2) for any nonempty subset of $A$, the product of its elements is not a perfect $p$-th power. What is the largest possible number of elements in $A$?

Problem 30 (02N01). What is the smallest positive integer $t$ such that there exist integers $x_1, x_2, \cdots, x_t$ with
\[ x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002}. \]

Problem 31 (02N02 - IMO 02/4). Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than $n^2$, and determine when it is a divisor of $n^2$.

Problem 32 (02N03). Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least $4^n$ divisors.

Problem 33 (02N04). Is there a positive integer $m$ such that the equation
\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c} \]
has infinitely many solutions in positive integers $a, b, c$?

Problem 34 (02N05). Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \cdots, a_n$ be integers, none of which is a multiple of $m^{n-1}$. Show that there exist integers $e_1, e_2, \cdots, e_n$, not all zero, with $|e_i| < m$ for all $i$, such that $e_1a_1 + e_2a_2 + \cdots + e_na_n$ is a multiple of $m^n$.

Problem 35 (02N06 - IMO 02/3). Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers $a$ such that
\[ \frac{a^n + a - 1}{a^n + a^2 - 1} \]
is itself an integer.

Problem 36 (01N01). Prove that there is no positive integer $n$ such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals $k$. 

Problem 29 (03N08). Let $p$ be a prime number and let $A$ be a set of positive integers that satisfies the following conditions: (1) the set of prime divisors of the elements in $A$ consists of $p-1$ elements; (2) for any nonempty subset of $A$, the product of its elements is not a perfect $p$-th power. What is the largest possible number of elements in $A$?
Problem 37 (01N02). Consider the system \( x + y = z + u, \) \( 2xy = zu \). Find the greatest value of the real constant \( m \) such that \( m \leq \frac{x}{y} \) for any positive integer solution \( (x, y, z, u) \) of the system, with \( x \geq y \).

Problem 38 (01N03). Let \( a_1 = 11^{11}, a_2 = 12^{12}, a_3 = 13^{13}, \) and \( a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4. \) Determine \( a_{14^{14}}. \)

Problem 39 (01N04). Let \( p \geq 5 \) be a prime number. Prove that there exists an integer \( a \) with \( 1 \leq a \leq p-2 \) such that neither \( a^{p-1} - 1 \) nor \( (a+1)^{p-1} - 1 \) is divisible by \( p^2 \).

Problem 40 (01N05 - IMO 01/6). Let \( a > b > c > d \) be positive integers and suppose that
\[
ac + bd = (b + d + a - c)(b + d - a + c).
\]
Prove that \( ab + cd \) is not prime.

Problem 41 (01N06). Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?

Problem 42 (00N01). Determine all positive integers \( n \geq 2 \) that satisfy the following condition: For all integers \( a \) and \( b \) relatively prime to \( n, \)
\[
a \equiv b \pmod{n}
\]
if and only if
\[
ab \equiv 1 \pmod{n}.
\]

Problem 43 (00N02). For every positive integers \( n \) let \( d(n) \) the number of all positive integers of \( n. \) Determine all positive integers \( n \) with the property: \( d^3(n) = 4n. \)

Problem 44 (00N03 - IMO 01/5). Does there exist a positive integer \( n \) such that \( n \) has exactly 2000 prime divisors and \( n \) divides \( 2^n + 1? \)

Problem 45 (00N04). Find all triplets of positive integers \( (a, m, n) \) such that \( a^m + 1 | (a + 1)^n. \)

Problem 46 (00N05). Prove that there exist infinitely many positive integers \( n \) such that \( p = nr, \) where \( p \) and \( r \) are respectively the semiperimeter and the inradius of a triangle with integer side lengths.
**Problem 47** (00N06). Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.

**Problem 48** (99N01 - IMO 99/4). Find all the pairs of positive integers \((x, p)\) such that \(p\) is a prime, \(x \leq 2p\) and \(x^{p-1}\) is a divisor of \((p-1)^2 + 1\).

**Problem 49** (99N02). Prove that every positive rational number can be represented in the form \(\frac{a^3 + b^3}{c^3 + d^3}\) where \(a, b, c, d\) are positive integers.

**Problem 50** (99N03). Prove that there exists two strictly increasing sequences \((a_n)\) and \((b_n)\) such that \(a_n(a_n + 1)\) divides \(b_n^2 + 1\) for every natural \(n\).

**Problem 51** (99N04). Denote by \(S\) the set of all primes such the decimal representation of \(\frac{1}{p}\) has the fundamental period divisible by 3. For every \(p \in S\) such that \(\frac{1}{p}\) has the fundamental period \(3r\) one may write
\[
\frac{1}{p} = 0.a_1a_2\cdots a_3a_1a_2\cdots a_3\cdots,
\]
where \(r = r(p)\); for every \(p \in S\) and every integer \(k \geq 1\) define \(f(k, p)\) by
\[
f(k, p) = a_k + a_{k+r(p)} + a_{k+2r(p)}
\]
a) Prove that \(S\) is infinite. b) Find the highest value of \(f(k, p)\) for \(k \geq 1\) and \(p \in S\).

**Problem 52** (99N05). Let \(n, k\) be positive integers such that \(n\) is not divisible by 3 and \(k \geq n\). Prove that there exists a positive integer \(m\) which is divisible by \(n\) and the sum of its digits in decimal representation is \(k\).

**Problem 53** (99N06). Prove that for every real number \(M\) there exists an infinite arithmetic progression such that:

1. each term is a positive integer and the common difference is not divisible by 10;
2. the sum of the digits of each term (in decimal representation) exceeds \(M\).