1 (Schur Theorem) Suppose the set $M = \{1, 2, \ldots, n\}$ is partitioned into t disjoint subsets $M_1, \ldots, M_t$. Show that if $n \geq \lfloor t! \cdot e \rfloor$ then at least one class M_z contains three elements x_i, x_j, x_k with the property that $x_i - x_j = x_k$.

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First Solution.

Fact 1. *Using Taylor Series approximation for the function $f(x) = e^x$ at point 0 for $x = 1$, we obtain the well-known identity*

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots,$$

hence

$$t! \cdot e = t! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{t!}\right) + \frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \ldots \tag{1}$$

Note that, for $t \geq 2$,

$$\frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \ldots < \frac{1}{t+1} + \frac{1}{(t+1)^2} + \frac{1}{(t+1)^3} + \ldots = -1 + \frac{1}{1 - \dfrac{1}{t+1}} = \frac{1}{t},$$

hence $S_t = \lfloor t! \cdot e \rfloor = t! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{t!}\right)$. It is easy to see that the sequence $(S_t)_{t \geq 0}$ satisfies the recurrence relation

$$S_t = tS_{t-1} + 1 \tag{2}$$

for $t \geq 1$, defining $S_0 = 1$.

Now we can proceed with the solution of the problem.

Assume no subset of the partition contains three elements a, b, c so that $a + b = c$. From the recurrence relation we have $t \not| S_t$ hence by Pigeonhole Principle, at least $\left\lfloor \frac{S_t}{t} \right\rfloor + 1 = S_{t-1} + 1$ elements of M are found in the same subset of the partition. Denote this subset by $M_1 = \{x_1, x_2, \ldots, x_k\}$ so that $x_1 < \ldots < x_k$, and $k \geq S_{t-1} + 1$. Consider the set $Y = \{y_1, \ldots, y_{k-1}\}$, defined by $y_i = x_{i+1} - x_1$. Clearly $|Y| = k - 1 \geq S_{t-1}$ and no element of Y is in M_1 (otherwise, if $y_i \in M_1$, then $y_i + x_1 = x_{i+1}$, contradiction). Consequently all elements of Y lie in the remaining $t-1$ subsets. Using similar arguments, at least $\left\lfloor \frac{k-1}{t-1} \right\rfloor \geq \left\lfloor \frac{S_{t-1} - 1}{t-1} \right\rfloor + 1 = S_{t-2} + 1$ elements of Y are found in the same subset from the partition of M. Without loss of generality, let M_2 be this subset. Then $M_2 = \{y_1, \ldots, y_s\} = \{x_2 - x_1, \ldots, x_{s+1} - x_1\}$, where $s \geq S_{t-2} + 1$. Because $y_i - y_1 = x_{i+1} - x_2$, we obtain $y_i - y_1 \notin M_1 \cup M_2$. Let $Z = \{y_2 - y_1, \ldots, y_s - y_1\} = \{x_3 - x_2, x_4 - x_2, \ldots, x_{s+1} - x_2\}$. Then the $|Z| \geq S_{t-2}$ elements of Z are in the remaining $t - 2$ subsets of the partition. By an easy induction, we get that the subset $M_i = \{x_i - x_{i-1}, x_{i+1} - x_{i-1}, \ldots, \} = \{y_{i-1} - y_{i-2}, y_i - y_{i-2}, \ldots\}$ of the partition contains at least $S_{t-i} + 1$ elements, using at the induction step the observation that the difference of any two elements of the set M_i, $i > 1$, is the difference of some 2 elements of each of the sets $M_1, \ldots, M_{i-1}$. Moreover for each $j < i$ there is an $z \in M_j$ so that for each $c \in M_i$, there is a $d \in M_j$ so that $c = d - z$.

In the end, the set M_t will contain at least $S_0 + 1 = 2$ elements. Assume $M_t = \{a, b\}$ with $a < b$. Then the number $b - a$ must be in one of the subsets $M_1, \ldots, M_{t-1}$. Assume $b - a \in M_i$.

But b and a, are, again by the construction of the sets (M_j), of the form $z_m - z_k$ and $z_n - z_k$, where $z_m, z_n, z_k \in M_i$. We obtained a contradiction because $(b-a)+z_n = (z_m - z_k) - (z_n - z_k) + z_n = z_m$. □

Second Solution. We use a theorem of Ramsey:

Theorem 1. *Let $a_1, \ldots, a_k \geq 1$ be positive inteers and $k \geq 2$. There exists a smallest positive integer $n = R_k(a_1, \ldots, a_k)$ so that for any coloring with k colors of the complete graph K_n there is an index i, $1 \leq i \leq k$ and a complete subgraph K_{a_i} of K_n with all edges of the same color.*

A proof of this theorem can be found in almost any book on Combinatorics or Graph Theory. Now we will show that

Proposition 1. $R_t\left(\underbrace{3, 3, \ldots, 3}_{t times}\right) \leq \lfloor t! \cdot e \rfloor + 1$

Proof of Proposition 1. We proceed by induction on $t \geq 2$. For $t = 2$ we easily get $R(3,3) = 6$. Indeed, $R(3,3) > 5$ as a regular pentagon having edges of one color, and diagonals of the other contains no monochromatic triangle. On the other side, every vertex of a K_6 has at least 3 neighbours to which it is joined by edges of the same color, say 1. If any of the edges between these three neighbours has color 1, we are done, otherwise they form a monochromatic triangle with edges of color 2.

Assume the statement true for some $t \geq 2$. Let $n = \lfloor t! \cdot e \rfloor$. We will show Proposition 1 for $t+1$. Let $m = \lfloor (t+1)! \cdot e \rfloor + 1$. Each vertex of K_m is endpoint for $m-1$ edges. Using Fact 1, we have $m - 1 = \lfloor (t+1)! \cdot e \rfloor = 1 + (t+1)\lfloor t! \cdot e \rfloor = 1 + (t+1)n$, so any vertex V of K_m has at least $n+1$ neighbours with which it is joined by edges of the same color, say color $t+1$. Consider the complete graph G formed by these $n+1$ vertices. If some vertices A, B of this graph are joined by an edge of color $t+1$, then A, B, V form a monochromatic triangle. Otherwise all edges of G have one of t colors. Since G has $n+1 = \lfloor t! \cdot e \rfloor + 1$ vertices, by the induction hypothesis, it has a monochromatic triangle. Consequently K_m has a monochromatic triangle, so $R_{t+1}\left(\underbrace{3, 3, \ldots, 3}_{t+1 \text{times}}\right) \leq \lfloor (t+1)! \cdot e \rfloor + 1$, and the induction step is over. □

The statement of Schur Theorem follows easily from Proposition 1. Indeed, let $n = \lfloor t! \cdot e \rfloor$. Now assign to the vertices of a complete graph with $n+1$ vertices K_{n+1} the numbers $1, 2, \ldots, n, n+1$. Color each edge (i,j) of K_{n+1} with the color c, where $|i-j| \in M_c$. By Proposition 1 $R_t(3, 3, \ldots, 3) \leq \lfloor t! \cdot e \rfloor + 1 = n+1$, hence K_{n+1} contains a monochromatic triangle. Let $x < y < z$ be the vertices of this monochromatic triangle. Then $y - x, z - x$ and $z - y$ belong to the same set M_i, for some $1 \leq i \leq t$. Since $(y-x) + (z-y) = (z-x)$ the proof of Schur's Theorem is completed. □

Remark 1 (Schur Number)**.** *The Schur Number $S(t)$ is defined as the largest positive integer n so that there exists a partition in t subsets of the set $\{1, 2, \ldots, n\}$, no subsets containing three integers x, y, z so that $x + y = z$ (x, y, z need not be different). As of now, only the first 4 exact values of the Schur Number are known, namely $S(1) = 1, S(2) = 4, S(3) = 13$ and $S(4) = 44$. We have proved* *that $S(t) \leq \lfloor t! \cdot e \rfloor - 1$. This upper bound can be slightly improved to $S(t) \leq \left\lfloor t! \left(e - \frac{1}{24} \right) \right\rfloor - 1$. From among the lower bounds, the following estimations are known: $S(t) \geq 2^t - 1$, $S(t) \geq \frac{3^t - 1}{2}$ and $S(t) \geq c \cdot 321^{\frac{t}{5}}$ for $t > 5$ and some constant c.*

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