(Schur Theorem) Suppose the set $M = \{1, 2, \ldots, n\}$ is partitioned into $t$ disjoint subsets $M_1, \ldots, M_t$. Show that if $n \geq \lfloor t! \cdot e \rfloor$ then at least one class $M_i$ contains three elements $x_i, x_j, x_k$ with the property that $x_i - x_j = x_k$.

First Solution.

Fact 1. Using Taylor Series approximation for the function $f(x) = e^x$ at point 0 for $x = 1$, we obtain the well-known identity

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots,$$

hence

$$t! \cdot e = t! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{t!}\right) + \frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \ldots$$

(1)

Note that, for $t \geq 2$,

$$\frac{1}{t+1} + \frac{1}{(t+1)(t+2)} + \ldots < \frac{1}{t+1} + \frac{1}{(t+1)^2} + \frac{1}{(t+1)^3} + \ldots = -1 + \frac{1}{1 - \frac{1}{t+1}} = \frac{1}{t},$$

hence $S_t = \lfloor t! \cdot e \rfloor = t! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{t!}\right)$. It is easy to see that the sequence $(S_t)_{t \geq 0}$ satisfies the recurrence relation

$$S_t = tS_{t-1} + 1$$

(2)

for $t \geq 1$, defining $S_0 = 1$.

Now we can proceed with the solution of the problem.

Assume no subset of the partition contains three elements $a, b, c$ so that $a + b = c$. From the recurrence relation we have $t \parallel S_t$, hence by Pigeonhole Principle, at least $\left\lfloor \frac{S_t}{t} \right\rfloor + 1 = S_{t-1} + 1$ elements of $M$ are found in the same subset of the partition. Denote this subset by $M_1 = \{x_1, x_2, \ldots, x_{k}\}$ so that $x_1 < \ldots < x_k$, and $k \geq S_{t-1} + 1$. Consider the set $Y = \{y_1, \ldots, y_{k}\}$, defined by $y_i = x_{i+1} - x_i$. Clearly $|Y| = k - 1 \geq S_{t-1}$ and no element of $Y$ is in $M_1$ (otherwise, if $y_i \in M_1$, then $y_i + x_1 = x_{i+1}$, contradiction). Consequently all elements of $Y$ lie in the remaining $t - 1$ subsets. Using similar arguments, at least $\left\lfloor \frac{k - 1}{t - 1} \right\rfloor \geq \left\lfloor \frac{S_{t-1} - 1}{t - 1} \right\rfloor + 1 = S_{t-2} + 1$ elements of $Y$ are found in the same subset from the partition of $M$. Without loss of generality, let $M_2$ be this subset. Then $M_2 = \{y_1, \ldots, y_s\} = \{x_2 - x_1, \ldots, x_{s+1} - x_1\}$, where $s \geq S_{t-2} + 1$. Because $y_i - y_1 = x_{i+1} - x_2$, we obtain $y_i - y_1 \notin M_1 \cup M_2$. Let $Z = \{y_2 - y_1, \ldots, y_s - y_1\} = \{x_3 - x_2, x_4 - x_2, \ldots, x_{s+1} - x_2\}$. Then the $|Z| \geq S_{t-2}$ elements of $Z$ are in the remaining $t - 2$ subsets of the partition. By an easy induction, we get that the subset $M_i = \{x_i - x_{i-1}, x_{i+1} - x_{i-1}, \ldots\} = \{y_{i-1} - y_{i-2}, y_{i-2} - y_{i-3}, \ldots\}$ of the partition contains at least $S_{t-i} + 1$ elements, using at the induction step the observation that the difference of any two elements of the set $M_i$, $i > 1$, is the difference of some 2 elements of each of the sets $M_1, \ldots, M_{t-1}$. Moreover for each $j < i$ there is an $z \in M_j$ so that for each $c \in M_i$, there is a $d \in M_j$ so that $c = d - z$.

In the end, the set $M_i$ will contain at least $S_0 + 1 = 2$ elements. Assume $M_i = \{a, b\}$ with $a < b$. Then the number $b - a$ must be in one of the subsets $M_1, \ldots, M_{t-1}$. Assume $b - a \in M_1$. In the end, the sets $M_i$ will contain at least $S_0 + 1 = 2$ elements. Assume $M_i = \{a, b\}$ with $a < b$. Then the number $b - a$ must be in one of the subsets $M_1, \ldots, M_{t-1}$. Assume $b - a \in M_1$.
We use a theorem of Ramsey:

Let \( S \) be the Schur Number.

The Schur Number are known, namely \( S(1) = 1, S(2) = 4, S(3) = 13 \) and \( S(4) = 44 \). We have proved

\[
R(3,3) = 6.
\]

\[
R(3,3,\ldots,3) = \left\lfloor (t+1)! \cdot e \right\rfloor + 1
\]

The statement of Schur Theorem follows easily from Proposition 1. Indeed, let \( n = \left\lfloor t! \cdot e \right\rfloor \). Now assign to the vertices of a complete graph with \( n+1 \) vertices \( K_{n+1} \) the numbers \( 1, 2, \ldots, n, n+1 \). Color each edge \((i,j)\) of \( K_{n+1} \) with the color \( c \), where \( |i-j| \in M_c \). By Proposition 1 \( R_t(3,3,\ldots,3) \leq \left\lfloor t! \cdot e \right\rfloor + 1 = n+1 \), hence \( K_{n+1} \) contains a monochromatic triangle. Let \( x < y < z \) be the vertices of this monochromatic triangle. Then \( y-x, z-x, \) and \( z-y \) belong to the same set \( M_i \), for some \( 1 \leq i \leq t \). Since \( (y-x)+(z-y) = (z-x) \) the proof of Schur’s Theorem is completed.

Remark 1 (Schur Number). The Schur Number \( S(t) \) is defined as the largest positive integer \( n \) so that there exists a partition in \( t \) subsets of the set \( \{1,2,\ldots,n\} \), no subsets containing three integers \( x, y, z \) so that \( x+y = z \) (\( x, y, z \) need not be different). As of now, only the first 4 exact values of the Schur Number are known, namely \( S(1) = 1, S(2) = 4, S(3) = 13 \) and \( S(4) = 44 \). We have proved
that \( S(t) \leq |t! \cdot e| - 1 \). This upper bound can be slightly improved to \( S(t) \leq \left\lfloor t! \left( e - \frac{1}{24} \right) \right\rfloor - 1 \).

From among the lower bounds, the following estimations are known: \( S(t) \geq 2^t - 1 \), \( S(t) \geq \frac{3^t - 1}{2} \) and \( S(t) \geq c \cdot 321^t \) for \( t > 5 \) and some constant \( c \).

References

6. A. Engel, Problem Solving Strategies, Chapter 4, The Box Principle.
13. L. Moser, An Introduction to the Theory of Numbers, Chapter 7, Combinatorial Number Theory


