1 Sequences of Consecutive Integers

1 A37 If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

PEN A37 A9 O51

A9 Prove that among any ten consecutive positive integers at least one is relatively prime to the product of the others.

O51 Prove the among 16 consecutive integers it is always possible to find one which is relatively prime to all the rest.

In the first part of this article, we shall show solutions for the problems. After that, historical background and connection between these problems will be shown. In the end, their generalizations and known results will be given.

Solution-A37 [SA]. We will assume the contrary, i.e. this product (denote it with A) can be a square. Amongst the numbers $n+1, n+2, \ldots, n+10$ at most 4 numbers are divisible by 5 or 7. Amongst the observed integers there are also at least 6 numbers of the form $2^\alpha 3^\beta c$ where all prime factors of c are greater than 10. It implies that c is a perfect square, so these 6 numbers have the form $2^\alpha 3^\beta k^2$. Now, the possible parity combinations of α, β are

1. α even, β even
2. α even, β odd
3. α odd, β even
4. α odd, β odd

Pigeonhole principle implies that at least two of these numbers have the same parity scheme:

1. In this case two of observed numbers have to be perfect squares, and since $(x+1)^2-x^2 = 2x+1$ one of these squares has to be $2^2, 3^2$ or 4^2. In each case A is not a perfect square.

2. In this case amogst the numbers $n+1, n+2, \ldots, n+10$ two have the form $3x^2$ and $3y^2$, and since for $x > y$ inequality $|3x^2-3y^2| \geq 3(y+1)^2-3y^2 = 3(2y+1)$ holds, so $y = 1$ which does not make A a perfect square. $\square$

Solution-A9 Tom Lovering [TL]. Clearly the only common prime factors amongst 10 consecutive positive integers will be $2, 3, 5, 7$.

5 of them will be divisible by 2, at least one of which must also be divisible by 3 and at least one of which must also be divisible by 5, and, if two of the numbers are divisible by 7, one of them will be even.

This leaves two multiples of 3, one multiple of 5, and one multiple of 7 unaccounted for, which makes 4 more of our integers.

But this still leaves one integer not divisible by $2, 3, 5, 7$, and therefore co-prime with all other integers of the set, and so coprime with their product. $\square$

Solution-O51 Tomek Kobos [TK]. Let A be the arbitrary set of 16 consecutive integers. It is clear that if p is a common prime factor of two elements of A then $p \in \{2, 3, 5, 7, 11, 13\}$. Thus, it is enough to show that we can find an element of A which is not divisible by any number from this set. Consider the following residues mod 30: $1, 7, 11, 13, 17, 19, 23, 29$. They are coprime with 30 and note that no matter how we choose the 16 consecutive residues there are always 4 of them belonging to this set. The difference between two of this numbers is never divisible by 7 and 13 and the only differences divisible by 11 are $23 - 1 = 22 = 29 - 7$ but this numbers are too far of each other to be in the same set of 16 consecutive integers. Hence among those 4 numbers there is at most one number divisible by 7, at most one number divisible by 11 and at most one number divisible by 13, so we can choose a number which is not divisible by any of them and since this number is also coprime to 30 we are done. $\square$

It is not hard to see the connection between A9 and O51, but how are they related to A37? Well, some of our readers may be tempted to use the result proven in A9 to prove the claim in problem A37. That is one of the methods Indian number theorist Subbayya Sivasankaranarayana Pillai used in his efforts to prove this general result:

Proposition 1.1 (Erdos-Selfridge Theorem)**.** *Product of consecutive integers is never a power.*

Pillai made some progress in that field, and the reader can see his conclusions in [P2] (remark: many interesting facts, theorems and their proofs can be found in papers listed in the ned of this article, and the reader is strongly encouraged to study them-it is trully

wonderful to find such pearls of number theory available online as public domain- to paraphrase Newton, it really helps us keep our balance on the giants' shoulders).
Still, Pillai's methods shown in the article mentioned above aren't efficient in this case. Still, in that time it was already proven that a product of at most 202 consecutive integers is not a square (proof given by Seimatsu Narumi in 1917). In 1939, Erdos proved the general claim that product of an arbitrary number of consecutive integers is not a perfect square. We ommited this proof here, but not due to its complexity, but due to its length. Still, the most general claim wasn't proven. Pillai's work on relative primality of consecutive integers had some interesting results apart from the 'product-power problem'. Pillai has first shown that the least number k for which a sequence of k consecutive integers without a number relatively prime to all others exists is 17.
The least example of such a sequence is $2184, 2185, \ldots, 2200$, and an infinite number of those sequences can be obtained by adding all numbers in the sequence a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$. Pillai conjectured ([P2]) that the following theorem holds:

Proposition 1.2. *For every integer $k > 16$ there exists a sequence of k consecutive integers without a number relatively prime to all others.*

The conjecture was later proven by both Alfred Brauer in [AB] and Pillai in [P3]. Both Pillai's and Brauer's proof are interesting as an introduction for a young number theorist to prime number distributions. For more information about finding such sequences, see [SS].

In 1975, Erdos and Selfridge finally published a proof that product of consecutive integers is never a power in [E3]. Due to its complexity, this proof has been omitted here. Contestants may find proofs of certain special cases of this theorem given in [TT] useful.
It seems like the story about these sequences is over, but that is not true. Propositions 1 and 2 have been generalized in various ways, and the mathematicians still seek for proofs of such generalizations. For instance in the work of Yair Caro, the relative primality in sequence 2 was replaced by a condition $\gcd(a, b) = d$, so the Proposition 2 (and its bound 17) was just made a special case for $d = 1$ (more about it in [S1],[S2],[IG]).
Proposition 2 has been generalized for arbitrary arythmetical progressions $a + nd$, which made the original theorem a special case sof $d = 1$. Another variation was made by replacing 'powers' with the so-called 'almost perfect powers'. More about it can be found in

T.N. Shorey's papers, available from [SB].

References

[Solutions of the problems]

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[Proofs of general theorems cited in the article]

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[Generalizations]

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[S2] N.Saradha, R. Thangadurai *Pillai's problem on consecutive integers*, Conference on Number Theory and Cryptography at HRI, Allahabad, 2007. http://www.math.tifr.res.in/~saradha/st-proc-r-f2.pdf

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